Abstract

Notes for MAA-5616 Measure Theory I, taught by Dr. Alexander Reznikov in Spring 2019. In these notes, $A \backslash B := A \cap B^c$. Also, the statement "$A \subset B$" does not necessarily mean that A is a proper subset of B. Homeworks have been included in these notes and are labelled as "Problem". Knowledge of basic set theory is assumed, as is familiarity with an introductory course on Real Analysis.

I accept sole responsibility for errors, of which I believe there will be many. Please feel free to offer feedback at amalik at math dot fsu dot edu

INTRODUCTION TO MEASURE THEORY

July 31, 2019

Contents

Syllabus

The purpose of this course is to introduce the notion of measure spaces and integrals against general measures; in particular, to construct the Lebesgue measure and Lebesgue Integral

Text book: Real Analysis by Royden and Fitzpatrick

Course Objectives: The purpose of this course is to introduce the notion of measure spaces and integrals against general measures; in particular, to construct the Lebesgue measure and Lebesgue integral.

Homeworks: Weekly graded homeworks will be given. Students are encouraged to collaborate on homeworks, but every student should write the final solutions on his or her own. It is crucial for passing the qualifying exam to understand every homework problem in all details.

Grading: There will be a midterm, a final exam, and weekly homeworks

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Syllabus Change Policy: Except for changes that substantially affect implementation of the evaluation (grading) statement, this syllabus is a guide for the course and is subject to change with advance notice.

1 Measure

1.1 General Measure Spaces

In general, a measure is a function applied on a collection of sets. Both the function and the domain have special properties. The ultimate idea is to define a function that somehow "measures" a decent set. This measurement may be in the form of length, volume, number of elements or other appropriate sense of measure. In general, the function itself must follow some commonsense properties, one of them being that the measure of union of disjoint sets must be the sum of measures.

To get other commonsense properties, the collection on which the function acts must have some structure to them. This is defined as follows: for any non-empty set X, the collection $\mathfrak{A}$ of subsets of X is called a **σ-algebra**, denoted by $(X, \mathfrak{A})$, a measure space, if

S1 $X \in \mathfrak{A}$

S2 $A, B \in \mathfrak{A} \implies A \backslash B \in \mathfrak{A}$

S3 $\{A_i : i \in \mathbb{N}\} \subset \mathfrak{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathfrak{A}$

Elements of a σ-algebra are called **measurable**.

For any σ-algebra $\mathfrak{A}$, $\varnothing \in \mathfrak{A}$ and $\mathfrak{A}$ is also closed under countable intersections.

Proof. For the first claim, let $A \in \mathfrak{A}$. Then, $A \backslash A \in \mathfrak{A}$ by **S2** so that $\varnothing \in \mathfrak{A}$.

Let $\{A_i : i \in \mathbb{N}\} \subset \mathfrak{A}$. Since $X \in \mathfrak{A}$ by **S1**, for each i, $X \backslash A_i = A_i^c \in \mathfrak{A}$. We also have

$$\bigcup_{i=1}^{\infty} A_i^c \in \mathfrak{A}$$

by **S3**. Since $X \in \mathfrak{A}$, again by **S1**, we have

$$X \backslash \bigcup_{i=1}^{\infty} A_i^c \in \mathfrak{A} \iff \bigcap_{i=1}^{\infty} A_i \in \mathfrak{A}$$

■

Definition 1 *Let $(X, \mathfrak{A})$ be a σ-algebra. A **measure** μ on $(X, \mathfrak{A})$ is a function $\mu : \mathfrak{A} \longrightarrow [0, \infty]$ such that*

M1 $\mu(\varnothing) = 0$, *and*

M2 For a disjoint family $\{A_i : i \in \mathbb{N}\} \subset \mathfrak{A}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

The triplet $(X, \mathfrak{A}, \mu)$ is then known as a **measurable space**. The property **M2** is known as the **countably additive property**.

Example 2 *Let X be any (countable) set. Define $\mu : 2^X \longrightarrow [0, \infty]$ as $\mu(A) = |A|$.*

Such a μ is called the **counting measure**. For our second example, we will need to be a little more precise since that concerns what's called the Borel σ-algebra on $\mathbb{R}$, $\mathfrak{L}$. $\mathfrak{L}$ is the smallest σ-algebra that contains all open sets of $\mathbb{R}$. Obviously, $\mathfrak{L}$ cannot contain only open sets A of $\mathbb{R}$ since, by definition, we must have $A \in \mathfrak{L} \implies A^c \in \mathfrak{L}$, where A^c "should" be open as well, but that is not the case. The σ-algebra is generated by adding all countable unions, countable intersections, and relative complements of all open (under the usual topology) subsets of $\mathbb{R}$. A process, called completion of a measurable space which we will see in §1.2, turns $(\mathbb{R}, \mathfrak{L}, \mathfrak{m}_1)$, the measurable space using Borel σ-algebra with the Lebesgue Measure, to $(\mathbb{R}, \mathfrak{M}_1, \mathfrak{m}_1)$, the Lebesgue Measurable Space. This fact will not be proved in these notes but the machinery to do so will be developed as we move on, as will many details of the latter space. And we have already moved ahead of ourselves. At any rate, a measure has the following properties:

Lemma 3 *Let X be a non-empty set and let μ be a measure on $(X, \mathfrak{A})$. Then,*

Finite Additivity *If $\{A_i : 1 \leq i \leq n\} \subset \mathfrak{A}$ is a collection of pairwise disjoint sets, then*

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

Monotonicity *$A, B \in \mathfrak{A}$ with $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$*

Complementation *$A, B \in \mathfrak{A}$ with $A \subset B$ and $\mu(A) < \infty \Longrightarrow \mu(B \backslash A) = \mu(B) - \mu(A)$*

Subadditivity *If $\{A_i : i \in \mathbb{N}\} \subset \mathfrak{A}$, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

Proof. Finite additivity

Let $A_{n+k} = \varnothing$ for $k = 1, 2, \ldots$. Then, $\{A_i : i \in \mathbb{N}\} \subset \mathfrak{A}$ is pairwise disjoint so that,

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

by **M2** and

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i) + 0 + 0 + \ldots = \sum_{i=1}^{n} \mu(A_i)$$

by **M1**

Monotonicity

Given that $A \subset B$, we can write $B = A \cup B \backslash A$. Then, by previous part, $\mu(B) = \mu(A \cup B \backslash A) = \mu(A) + \mu(B \backslash A)$. Since $B \backslash A \in \mathfrak{A}$ by **S2**, $\mu(B \backslash A) \geq 0$. Thus, $\mu(A) \leq \mu(B)$

Complementation

Proceeding as above, we have $\mu(B) = \mu(A) + \mu(B \backslash A)$ but now we can subtract the real number $\mu(A)$ on both sides as we have $\mu(A) < \infty$.

Subadditivity

Let $B_1 = A_1$ and

$$B_n = A_n \backslash \bigcup_{i=1}^{n-1} A_i$$

Then, the family $\{B_i : i \in \mathbb{N}\}$ is pairwise disjoint: assume that $i < j$ (the argument for $i > j$ is similar). Then,

$$\begin{aligned} B_i \cap B_j &= \left(A_i \backslash \bigcup_{k=1}^{i-1} A_k \right) \cap \left(A_j \backslash \bigcup_{m=1}^{j-1} A_m \right) \\ &= A_i \cap \bigcap_{k=1}^{i-1} A_k^c \cap A_j \cap \bigcap_{m=1}^{j-1} A_m^c = \varnothing \end{aligned}$$

since $A_i \cap A_{m_0}^c = \varnothing$ for some m_0 with $1 \leq m_0 = i < j$ (and, obviously, $A_i \cap A_i^c = \varnothing$). Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} B_i \right) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

since $B_i \subset A_i$ for each i. We will be done if we can prove that

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

Since $B_i \subset A_i$ for each i, we must have

$$\bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} A_i$$

For the converse, let

$$x \in \bigcup_{i=1}^{\infty} A_i$$

Define $I = \{j : x \in A_j\} \subset \mathbb{N}$. By the well-ordering principle, I has a least element, say m. Then,

$$x \in B_m = A_m \backslash \bigcup_{i=1}^{m-1} A_i$$

so that both inclusions hold. ■

The following properties mimic continuity properties.

Theorem 4 *Let $\{A_i : i \in \mathbb{N}\}$ be a family of measurable sets. If this family is increasing and nested, that is, we have a sequence of sets $A_1 \subset A_2 \subset ...$, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$$

In this case, μ is said to be ***continuous from below****. On the other hand, if we have a nested decreasing sequence $A_1 \supset A_2 \supset ...$ and $\mu(A_1) < \infty$, then*

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$$

In this case, μ is said to be ***continuous from above.***

Proof. For first part, let $E_1 = A_1$ and $E_n = A_n \backslash A_{n-1}$. This family is measurable by **S2** and disjoint: WLOG assume that $i < j$. Then, $E_i \cap E_j = A_i \cap A_{i-1}^c \cap A_j \cap A_{j-1}^c$

$= A_i \cap A_j \cap (A_{i-1} \cup A_{j-1})^c$

$= A_i \cap A_{j-1}^c = \varnothing$ because $i \leq j - 1$

Moreover,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} E_i$$

One side of the inclusion is obtained by observing that $E_i \subset A_i$ for each i, which gives us

$$\bigcup_{i=1}^{\infty} A_i \supset \bigcup_{i=1}^{\infty} E_i$$

For the reverse inclusion, let

$$x \in \bigcup_{i=1}^{\infty} A_i$$

Define $I = \{j : x \in A_j\} \subset \mathbb{N}$. By the well-ordering principle, I has a least element, say m. Then, $x \in E_m = A_m \backslash A_{m-1}$ with $A_0 := \varnothing$, so that

$$x \in \bigcup_{i=1}^{\infty} E_i$$

Then,

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i \backslash A_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (\mu(A_i) - \mu(A_{i-1}))
\end{aligned}$$

The last step is possible if we assume that $\mu(A_i) < \infty$ for each i, otherwise we would have $\infty = \infty$, so that equality still holds and the proof withstands scrutiny. Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n) - \mu(A_0)$$

Since we've assumed that $A_0 = \varnothing$, we have $\mu(A_0) = \varnothing$ and we're done.

For the second part, let $B_1 = \varnothing$, $B_2 = A_1 \backslash A_2$, $B_3 = A_1 \backslash A_3$, ... Then, B_i is an increasing sequence of sets: $B_i = A_1 \backslash A_i$ and $B_{i+1} = A_1 \backslash A_{i+1}$. Since $A_{i+1} \subset A_i$, we have $A_1 \backslash A_i \subset A_1 \backslash A_{i+1}$, or that $B_i \subset B_{i+1}$. Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right) \tag{1}$$

To finish the proof, we first show that

$$\bigcup_{i=1}^{\infty} B_i = A_1 \backslash \bigcap_{i=1}^{\infty} A_i$$

To see this, note that $B_1 = A_1 \backslash A_1 = \varnothing$. $B_2 = B_1 \cup B_2 = A_1 \backslash A_2 = A_1 \backslash (A_1 \cap A_2)$. This takes care of the base step for induction. Now, let

$$\bigcup_{i=1}^{k} B_i = A_1 \backslash \bigcap_{i=1}^{k} A_i$$

Then,

$$\begin{aligned} \bigcup_{i=1}^{k+1} B_i &= B_{k+1} = A_1 \backslash A_{k+1} \\ &= A_1 \backslash \bigcap_{i=1}^{k+1} A_i \end{aligned}$$

We can therefore have

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(A_1) - \mu\left(\bigcap_{i=1}^{\infty} A_i\right) \tag{2}$$

Equating (1) and (2), we get

$$\mu(A_1) - \lim_{n\to\infty} \mu(A_n) = \mu(A_1) - \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$$

Since $\mu(A_1) < \infty$, then we can cancel this real number from both sides to get

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$$

■

The converse also holds under different assumptions:

Problem 5 *Let X be a non-empty set and let $(X, \mathfrak{A})$ be a σ-algebra. Let $\mu : \mathfrak{A} \longrightarrow [0, \infty]$ be finitely additive and, for every nested increasing sequence $A_1 \subset A_2 \subset ...$ in $\mathfrak{A}$,*

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \lim_{n \to \infty} \mu(A_n)$$

Then, μ is a measure.

Solution 6 ***M1*** *Let $A \in \mathfrak{A}$ such that $\mu(A) < \infty$. Then, $\mu(A \cup \varnothing) = \mu(A) = \mu(A) + \mu(\varnothing)$ since A and $\varnothing$ are disjoint, so that $\mu(\varnothing) = 0$.*

M2 *Let $\{B_i : i \in \mathbb{N}\}$ be a sequence of disjoint sets. We need to show that*

$$\mu \left(\bigcup_{i=1}^{\infty} B_i \right) = \sum_{i=1}^{\infty} \mu(B_i)$$

Define

$$A_n = \bigcup_{i=1}^{n} B_i$$

with $A_1 = B_1$. Then,

$$A_n = \bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{n+1} B_i = A_{n+1}$$

gives us a family $\{A_i : i \in \mathbb{N}\}$ of an increasing sequence in $\mathfrak{A}$ so that

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \mu \left(\bigcup_{i=1}^{n} B_i \right) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i)$$

by finite additivity of μ so that

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(B_i)$$

It remains to show that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

Let

$$x \in \bigcup_{i=1}^{\infty} A_i$$

Then, $\exists j$ such that

$$x \in A_j = \bigcup_{i=1}^{j} B_i \implies x \in \bigcup_{i=1}^{\infty} B_i$$

Thus,

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} B_i$$

Conversely, since $B_i \subset A_i$ for each i, it follows that

$$\bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} A_i$$

and, therefore

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(B_i\right)$$

Problem 7 *Let X be a non-empty set and let $(X, \mathfrak{A})$ be a σ-algebra. Let $\mu : \mathfrak{A} \longrightarrow [0, \infty]$ be finitely additive. Then, μ is a measure if for every nested decreasing sequence $A_1 \supset A_2 \supset ...$ in $\mathfrak{A}$ with $\mu(A_1) < \infty$,*

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu\left(A_n\right)$$

Solution 8 *Let $\{B_i : i \in \mathbb{N}\}$ be family of disjoint sets. Let*

$$A_n = \bigcup_{i=1}^{n} B_i$$

Then, $A_n \subset A_{n+1}$ so that we have a nested decreasing sequence $A_{n+1}^c \subset A_n^c$. Let us first show that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

To this end, we first note that

$$\forall x \in \bigcup_{i=1}^{\infty} A_i, \exists j \text{ such that } x \in A_j = \bigcup_{i=1}^{j} B_i \subset \bigcup_{i=1}^{\infty} B_i$$

Conversely,

$$x \in \bigcup_{i=1}^{\infty} B_i \Longrightarrow x \in B_j \text{ for some } j \text{ and } B_j \subset \bigcup_{i=1}^{j} B_i = A_j \subset \bigcup_{i=1}^{\infty} A_i$$

It follows that

$$\bigcap_{i=1}^{\infty} A_i^c = \bigcap_{i=1}^{\infty} B_i^c \Longrightarrow \mu\left(\bigcap_{i=1}^{\infty} B_i^c\right) = \mu\left(\bigcap_{i=1}^{\infty} A_i^c\right) = \lim_{n \to \infty} \mu\left(A_n^c\right)$$

Since μ is finitely additive, we must have $\mu(X) = \mu(C \cup C^c) = \mu(C) + \mu(C^c)$. That is, $\mu(C) = \mu(X) - \mu(C^c)$. Of course we can only say this, provided that

$\mu(C^c) < \infty$, which we will take the liberty of assuming (in the other case, the equality still holds). Moreover, we can denote

$$\bigcap_{i=1}^{\infty} A_i^c = A^c$$

Then,

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu(X) - \mu(A^c) = \mu(X) - \mu\left(\bigcap_{i=1}^{\infty} A_i^c\right) = \mu(X) - \lim_{n\to\infty} \mu(A_n^c)$$

That is,

$$\mu\left(\left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) = \mu\left(\left(\bigcup_{i=1}^{\infty} B_i\right)^c\right) = \mu(X) - \lim_{n\to\infty} \mu(A_n) = \mu(X) - \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i)$$

Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i)$$

Definition 9 *We say that a property holds* ***almost everywhere*** *or* ***for almost all*** *$x \in X$ if the measure of the set for which the property does not hold is zero.*

Lemma 10 (Borel-Cantelli) *If we have a family of measurable sets $\{E_i : i \in \mathbb{N}\}$ and their union has finite measure, i.e.,*

$$\sum_{i=1}^{\infty} \mu(E_i) < \infty,$$

then almost every $x \in X$ belongs to at most finitely many $E_j's$.

Proof. We need to show that the set $F = \{x \in X : x \text{ belongs to infinitely many } E_j\text{'s}\}$ is measurable and has zero measure. Let $x \in F$. Then, for every $N \in \mathbb{N}$,

$$x \in \bigcup_{n=N}^{\infty} E_n$$

Since this is true for every N, then

$$x \in \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n$$

Conversely,

$$x \in \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n \implies x \in F$$

Thus,

$$F = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n$$

so that F is measurable. Now, note that

$$\bigcup_{n=1}^{\infty} E_n \supset \bigcup_{n=2}^{\infty} E_n \supset \bigcup_{n=3}^{\infty} E_n \ldots$$

Then, by continuity from above, we have that

$$\mu(F) = \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} E_i\right) = \lim_{N\to\infty} \mu\left(\bigcup_{i=N}^{\infty} E_i\right) \le \lim_{N\to\infty} \sum_{i=N}^{\infty} \mu(E_i) = 0$$

because the union of $\{E_i : i \in \mathbb{N}\}$ has finite measure. ■

1.2 Completion of Measure

Let $(X, \mathfrak{A}, \mu)$ be a measure space and let

$$\mathfrak{M} = \{E \subset X : E = A \cup B \text{ s.t. } A \in \mathfrak{A} \wedge B \subset C : \mu(C) = 0\}$$

That is, we add sets to a collection with measure zero. Then, $\mathfrak{M}$ is a σ-algebra.
Proof. We need to show that (a) $X \in \mathfrak{M}$, (b) if $E, F \in \mathfrak{M}$, then $E \backslash F \in \mathfrak{M}$ and (c) $\mathfrak{M}$ is closed under countable unions

(a) Note that for each $A \in \mathfrak{A}$, $A = A \cup \varnothing$ and $\varnothing \subset \varnothing$ and $\mu(\varnothing) = 0$, so that $A \in \mathfrak{M}$. Thus, $\mathfrak{A} \subset \mathfrak{M}$ so that $X, \varnothing \in \mathfrak{M}$.

(b) We will first show that for any $E \in \mathfrak{M}$, $E^c \in \mathfrak{M}$ and then show that $\mathfrak{M}$ is closed under intersection. These two facts together will show that for any E, F in $\mathfrak{M}$, $E \backslash F = E \cap F^c \in \mathfrak{M}$.

Let $E \in \mathfrak{M}$ where $E = A \cup B$ with $A \in \mathfrak{A}$ and $B \subset C$ such that $\mu(C) = 0$. Note that $C \subset B$ implies $C^c \subset B^c$ so that $C^c \cup B^c = B^c$ and that

$B^c = X \cap B^c = (C^c \cup C) \cap (C^c \cup B^c) = C^c \cup (C \cap B^c)$.

Then, $E^c = A^c \cap B^c = A^c \cap (C^c \cup (C \cap B^c))$

$= (A^c \cap C^c) \cup (A^c \cap C \cap B^c)$

Now, since $A, C \in \mathfrak{A}$, then $A^c, C^c \in \mathfrak{A}$ and so $A^c \cap C^c \in \mathfrak{A}$. Also, by definition, $A^c \cap C \cap B^c \subset C$ and by assumption $\mu(C) = 0$ so that $(A^c \cap C^c) \cup (A^c \cap C \cap B^c) = E^c \in \mathfrak{M}$.

Next, let Let $E, F \in \mathfrak{M}$. Then, $\exists A_1, A_2, C_1, C_2 \in \mathfrak{A}$ such that $E = A_1 \cup B_1$, $F = A_2 \cup B_2$ and $B_i \subset C_i$ such that $\mu(C_i) = 0$ for $i = 1, 2$. Then, $E \cap F = (A_1 \cup B_1) \cap (A_2 \cup B_2)$

$= (A_1 \cap (A_2 \cup B_2)) \cup (B_1 \cap (A_2 \cup B_2))$

$= (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_1 \cap B_2)$

Now, $A_1 \cap A_2 \in \mathfrak{A}$ and $A_1 \cap B_2 \subset B_2 \subset C_2$, $B_1 \cap A_2 \subset B_1 \subset C_1$ and $B_1 \cap B_2$ is a subset of C_1 (and C_2). Thus, $(A_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_1 \cap B_2) \subset C_1 \cup C_2$. Note that $\mu(C_1 \cup C_2) \le \mu(C_1) + \mu(C_2) = 0 + 0 = 0$ so that $\mu(C_1 \cup C_2) = 0$ since $\mu(A) \ge 0$ for all $A \in \mathfrak{A}$. Thus, $E \cap F \in \mathfrak{M}$.

We can now assert that for any $E, F \in \mathfrak{M}$, $E \backslash F = E \cap F^c \in \mathfrak{M}$.

(c) Assume that $\{E_i : i \in \mathbb{N}\} \subset \mathfrak{M}$ is a family of disjoint sets, where, for each i, $E_i = A_i \cup B_i$ where $A_i \in \mathfrak{A}$ and $B_i \subset C_i$ such that $\mu(C_i) = 0$. Then,

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A_i \cup B_i) = \bigcup_{i=1}^{\infty} A_i \cup \bigcup_{i=1}^{\infty} B_i$$

Again,

$$\bigcup_{i=1}^{\infty} A_i \in \mathfrak{A} \text{ and } \bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} C_i$$

and that

$$\mu\left(\bigcup_{i=1}^{\infty} C_i\right) \leq \sum_{i=1}^{\infty} \mu(C_i) = 0 \implies \mu\left(\bigcup_{i=1}^{\infty} C_i\right) = 0 \implies \bigcup_{i=1}^{\infty} E_i \in \mathfrak{M}$$

furnishing a proof of the third claim. ■

For such E, define $\overline{\mu} : \mathfrak{M} \longrightarrow [0, \infty]$ by $\overline{\mu}(E) = \mu(A)$. This is a well-defined measure on $\mathfrak{M}$.

Proof. To show that $\overline{\mu}$ is well defined, assume that $E_1 = A_1 \cup B_1$ and $E_2 = A_2 \cup B_2$ with $B_1 \subset C_1$, $B_2 \subset C_2$ and $\mu(C_i) = 0$ for $i = 1, 2$. We need to show that $E_1 = E_2 \implies \overline{\mu}(E_1) = \overline{\mu}(E_2)$. Now, $E_1 = E_2$

$\implies A_1 \cup B_1 = A_2 \cup B_2$

$\implies A_1 \cup B_1 \cup C_1 \cup C_2 = A_2 \cup B_2 \cup C_1 \cup C_2$

$\implies A_1 \cup C_1 \cup C_2 = A_2 \cup C_1 \cup C_2$ since $B_i \subset C_i$ for $i = 1, 2$.

Now, since μ is measure on $\mathfrak{A}$, $A_1 \cup C_1 \cup C_2 = A_2 \cup C_1 \cup C_2 \in \mathfrak{A}$, $A_i \in \mathfrak{A}$ and $A_1 \subset A_1 \cup C_1 \cup C_2$, then $\mu(A_1) \leq \mu(A_1 \cup C_1 \cup C_2) = \mu(A_2 \cup C_1 \cup C_2) \leq \mu(A_2) + \mu(C_1) + \mu(C_2) = \mu(A_2) + 0 + 0 = \mu(A_2)$.

Similarly, $\mu(A_2) \leq \mu(A_2 \cup C_1 \cup C_2) = \mu(A_1 \cup C_1 \cup C_2) \leq \mu(A_1)$. Thus, $\mu(A_1) = \mu(A_2)$, which means that $\overline{\mu}(E_1) = \overline{\mu}(E_2)$

We now show that $\overline{\mu}$ is a measure.

M1 Since $\mathfrak{A} \subset \mathfrak{M}$ by (a) above, then $\overline{\mu}|_{\mathfrak{A}} = \mu$, by definition. Thus, $\overline{\mu}(\varnothing) = \mu(\varnothing) = 0$.

M2 Let $\{E_i : i \in \mathbb{N}\} \subset \mathfrak{M}$, where $E_i = A_i \cup B_i$ for each i, be a pairwise disjoint family of $\mathfrak{M}$-measurable subsets. Then,

$$\overline{\mu}\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \overline{\mu}(E_i)$$

making $\overline{\mu}$ a bona fide measure. ■

Problem 11 *Let $(X, \mathfrak{A}, \mu)$ be a measure space, $E \in \mathfrak{A}$ and $\mu(E) > 0$. Define $\mathfrak{A}_E := \{A \subset E : A \in \mathfrak{A}\}$. Show that that $\mathfrak{A}_E$ is a σ-algebra on E and the restriction of μ to E is a measure.*

Solution 12 *Since $E \subset E$, $E \in \mathfrak{A}_E$. Let $A, B \in \mathfrak{A}_E$. Then, $A \subset E$ and $A, B \in \mathfrak{A}$ (so $A \backslash B \in \mathfrak{A}$). Since $A \backslash B \subset A \subset E$, we have that $A \backslash B \subset E$. Thus, $A \backslash B \in \mathfrak{A}_E$. Finally, let $\{E_i : i \in \mathbb{N}\} \subset \mathfrak{A}_E$. Then, for each i, $E_i \subset E$ and $\{E_i : i \in \mathbb{N}\} \subset \mathfrak{A}$. These two facts imply that*

$$\bigcup_{i=1}^{\infty} E_i \subset E$$

by properties of sets and, respectively,

$$\bigcup_{i=1}^{\infty} E_i \in \mathfrak{A}$$

since $\mathfrak{A}$ is a σ-algebra. Thus

$$\bigcup_{i=1}^{\infty} E_i \in \mathfrak{A}_E$$

and $\mathfrak{A}_E$ is a σ-algebra. Let $\mu|_E = \nu$. Then, $\nu(\varnothing) = \mu|_E(\varnothing) = 0 = \mu(\varnothing) = 0$ since $\varnothing \in \mathfrak{A}$. Now, let $\{E_i : i \in \mathbb{N}\} \subset \mathfrak{A}_E$ be pairwise disjoint. Then,

$$\bigcup_{i=1}^{\infty} E_i \in \mathfrak{A}_E \subset \mathfrak{A}$$

so that

$$\begin{aligned} \nu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu|_E\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \\ &= \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} \mu|_E(E_i) = \sum_{i=1}^{\infty} \nu(E_i) \end{aligned}$$

1.3 Construction of Measures

There is a canonical way to build measures. For that, we need a little machinery.

Definition 13 *Let $S \subset 2^X$ and $\mu : S \longrightarrow [0, \infty]$ be a function. μ is said to be* ***countably monotone*** *if for every $E \in S$, and for every $\{E_i : i \in \mathbb{N}\} \subset S$, we have*

$$E \subset \bigcup_{i=1}^{\infty} E_i \implies \mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

Definition 14 *Let $S \subset 2^X$ and $\mu : S \longrightarrow [0, \infty]$ be a function. μ is said to be* ***countably subadditive*** *if for every $\{A_i : i \in \mathbb{N}\} \subset S$,*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

Definition 15 *Let $S \subset 2^X$ and $\mu : S \longrightarrow [0, \infty]$ be a function. μ is said to be a* ***pre-measure*** *if*

P1 $\mu(\varnothing) = 0$;

P2 μ *is finitely additive; and*

P3 μ *is countably monotone*

Definition 16 *An* ***outer measure****, $\mu^* : 2^X \longrightarrow [0, \infty]$, is a function definable on every subset of a non-empty set X such that*

O1 $\mu^*(\varnothing) = 0$

O2 μ^* *is countably monotone.*

Lemma 17 *μ^* is countably subadditive and, therefore, finitely subadditive.*

Proof. Let $\{A_i : i \in \mathbb{N}\} \subset 2^X$. Then,

$$A = \bigcup_{i=1}^{\infty} A_i \implies A \subset \bigcup_{i=1}^{\infty} A_i$$

so that

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

■

Example 18 *The trivial measure $\mu^*(A) = 0 \ \forall A$. The counting measure is clearly another example.*

Here is another interesting example.

Lemma 19 *Let $S \subset 2^X$ and let $\mu : S \longrightarrow [0, \infty]$ be a function (we do not even need μ to be a pre-measure!). Define $\mu^* : 2^X \longrightarrow [0, \infty]$ by $\mu^*(\varnothing) = 0$ and*

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i \wedge E_i \in S \forall i\right\}$$

Then, μ^* is an outer measure.

Proof. We already have $\mu^*(\varnothing) = 0$ so that **O1** is trivially satisfied. Thus, to prove that μ^* is an outer measure, we only need to prove **O2** – countable monotonicity from covering of E by $\{E_i : i \in \mathbb{N}\}$:

$$E \subset \bigcup_{i=1}^{\infty} E_i \implies \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$$

If for some i, $\mu^*(E_i) = \infty$, we are done. Assume $\mu^*(E_i) < \infty$ for all i. Let $\epsilon > 0$. For every i, by properties of infimum, we can cover E_i by a family $\{E_i^k : k \in \mathbb{N}\} \subset S$ such that

$$\mu^*(E_i) \geq \sum_{k=1}^{\infty} \mu(E_i^k) - \frac{\epsilon}{2^i} \implies E \subset \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{k,i=1}^{\infty} E_i^k$$

That is, E is covered by $\{E_i^k : k \in \mathbb{N}\}$, a countable cover. By definition of μ^*,

$$\mu^*(E) = \inf_{E_k} \sum_{i,k=1}^{\infty} \mu(E_i^k) \leq \sum_{i,k=1}^{\infty} \mu(E_i^k)$$

Since all the terms of the infinite sum are positive, we can re-arrange the terms to get

$$\sum_{i,k=1}^{\infty} \mu(E_i^k) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \mu(E_i^k) \leq \sum_{i=1}^{\infty} \left(\mu^*(E_i) + \frac{\epsilon}{2^j}\right) = \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon$$

Now, we can pass the limit to ϵ to get

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$$

■

The idea over here is to take a covering and bring it down to a different family within S.

Notice that the definition of the outer measure does not assume that the domain is a σ-algebra. With slight modifications, however, a subdomain of μ^* forms a σ-algebra: let us first call a set $E \subset X$ **measurable with respect to** μ^* if, for any $A \subset X$, we have $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Note that $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ is always true because $A = A \cap X = A \cap (E \cup E^c) = (A \cap E) \cup (A \cap E^c)$ and because μ^* is finitely subadditive.

Proposition 20 *Let $E \subset X$. If $\mu^*(E) = 0$, then E is measurable with respect to μ^**

Proof. Let $A \subset X$. Since $\mu^*(A \cap E) \leq \mu^*(E)$, we must have $\mu^*(A \cap E) = 0$. Again, by monotonicity, $\mu^*(A) \geq \mu^*(A \cap E^c) = \mu^*(A \cap E^c) + \mu^*(A \cap E)$. ■

Such a set, which we will call $\mathfrak{M}$, of μ^*-measurable sets form a σ-algebra. The proof of this fact is broken into a bunch of lemmas. First, note that $\mu^*(A) = \mu^*(A \cap \varnothing) + \mu^*(A \cap \varnothing^c)$ so that $\mu^*(\varnothing) = 0$ so that $\varnothing$ is measurable with respect to μ^* and hence $\varnothing \in \mathfrak{M}$. Also note that if E is measurable with respect to μ^*, then E^c is measurable with respect to μ^*, by simple commutativity of the real numbers under addition.

Lemma 21 *If E_1 and E_2 are measurable with respect to μ^*, then $E_1 \cup E_2$ is measurable.*

Proof. If E_1 is measurable, then for any A, $\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)$. Also, again, since E_2 is measurable, then letting $B = A \cap E_1^c$, we have, for all B, $\mu^*(B) = \mu^*(B \cap E_2) + \mu^*(B \cap E_2^c)$. Thus, $\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)$
$= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c)$
$= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap (E_1 \cup E_2)^c)$

If A, E_1 and E_2 have a non-trivial intersection, then $A \cap (E_1 \cup E_2) = A \cap X \cap (E_1 \cup E_2) = A \cap (E_1 \cup E_1^c) \cap (E_1 \cup E_2)$
$= (A \cap E_1) \cup (A \cap E_1^c) \cap (E_1 \cup E_2)$
$= (A \cap E_1) \cup (A \cap E_1^c \cap E_1) \cup (A \cap E_1^c \cap E_2)$
$= (A \cap E_1) \cup (A \cap E_1^c \cap E_2)$ so that $\mu^*(A \cap (E_1 \cup E_2)) \leq \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2)$ by finite subadditivity, with equality if E_1 and E_2 cover A. Thus, $\mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c)$
$\leq \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap (E_1 \cup E_2)^c)$
$= \mu^*(A)$

Thus, $\mu^*(A) = \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c)$. ■

Lemma 22 *If E_1 and E_2 are measurable with respect to μ^*, then $E_1 \backslash E_2$ is measurable with respect to μ^*.*

Proof. We've already proved that for $E_1, E_2 \in \mathfrak{M}$, $E_1 \cup E_2 \in \mathfrak{M}$ and that $E_1^c, E_2^c \in \mathfrak{M}$. Since $E_1 \backslash E_2 = E_1 \cap E_2^c = (E_1^c \cup E_2)^c$, we have that $E_1 \backslash E_2 \in \mathfrak{M}$ ■

Lemma 23 *If E_1 and E_2 are measurable with respect to μ^* and disjoint, then, for every A, $\mu^*(A \cap (E_1 \cup E_2)) = \mu^*(A \cap E_1) + \mu^*(A \cap E_2)$*

Proof. Since E_2 is measurable, we have $\mu^*(A \cap (E_1 \cup E_2)) = \mu^*(A \cap (E_1 \cup E_2) \cap E_2) + \mu^*(A \cap (E_1 \cup E_2) \cap E_2^c)$
$= \mu^*(A \cap ((E_1 \cap E_2) \cup (E_2 \cap E_2))) + \mu^*(A \cap ((E_1 \cap E_2^c) \cup (E_2 \cap E_2^c)))$

Since $E_1 \cap E_2 = \varnothing$, we have $\mu^*(A \cap (\varnothing \cup E_2)) + \mu^*(A \cap ((E_1 \cap E_2^c) \cup \varnothing))$
$= \mu^*(A \cap E_2) + \mu^*(A \cap (E_1 \cap E_2^c))$

Again, since $E_1 \cap E_2 = \varnothing$, we must have that $E_1 \subset E_2^c$. Thus, $E_1 \cap E_2^c = E_1$ so that $\mu^*(A \cap (E_1 \cup E_2)) = \mu^*(A \cap E_2) + \mu^*(A \cap E_1)$ ■

By induction, the above holds for n sets.

Lemma 24 *If $\{E_i : i \in \mathbb{N}\}$ are measurable with respect to μ^*, then so is their union.*

Proof. Let

$$\widetilde{E}_k = E_k \backslash \bigcup_{i=1}^{k-1} E_i \text{ and } F_n = \bigcup_{i=1}^{n} \widetilde{E}_i = \bigcup_{i=1}^{n} E_i$$

F_n is measurable because finite unions are measurable. Now set

$$E = \bigcup_{i=1}^{\infty} E_i$$

We need to prove that $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ i.e., E is μ^*-measurable. We know that $F_n \subset E$ for any n so that $E^c \subset F_n^c$. Therefore,

$$\begin{aligned}
\mu^*(A) &= \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c) \\
&\geq \mu^*(A \cap F_n) + \mu^*(A \cap E^c) \\
&= \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} \widetilde{E}_i\right)\right) + \mu^*(A \cap E^c)
\end{aligned}$$

Since $\widetilde{E}_i$'s are disjoint, we then have, by **Lemma 23**,

$$= \sum_{i=1}^{n} \mu^*\left(A \cap \widetilde{E}_i\right) + \mu^*(A \cap E^c)$$

Now, we also know that

$$\bigcup_{i=1}^{\infty} \widetilde{E}_i = \bigcup_{i=1}^{\infty} E_i = E \Longrightarrow A \cap E = \bigcup_{i=1}^{\infty} \left(A \cap \widetilde{E}_i\right)$$

so that, by countable subadditivity, we have

$$\mu^*(A \cap E) \leq \sum_{i=1}^{\infty} \mu^*\left(A \cap \widetilde{E}_i\right) \implies \mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \sum_{i=1}^{\infty} \mu^*\left(A \cap \widetilde{E}_i\right) + \mu^*(A \cap E^c)$$

Thus,

$$\begin{aligned}
\mu^*(A) &\geq \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + \mu^*(A \cap E^c) \\
&= \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*(A \cap E^c) \\
&\geq \mu^*(A \cap E) + \mu^*(A \cap E^c)
\end{aligned}$$

The other inequality holds trivially. ■

Thus, the collection of all measurable sets $\mathfrak{M}$ under μ^* forms a σ-algebra. It can now be shown that $\overline{\mu} = \mu^*|_{\mathfrak{M}}$ forms a (complete) measure on $\mathfrak{M}$. For this to be valid, we need to prove that a family of pairwise disjoint sets $\{E_i : i \in \mathbb{N}\}$,

$$\overline{\mu}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \overline{\mu}(E_i)$$

One inequality of this holds always because of countably monotone property. For the other side, by **Lemma 23**, for a (finite) family of pairwise disjoint sets $\{E_i\}$, for any A,

$$\mu^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$$

Substituting $A = X$, we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{n} \mu^*(E_i)$$

so that we can replace n with ∞, which is what is needed. Thus, $\overline{\mu}$ is a measure.

To prove that $\overline{\mu}$ is complete (given a measurable subset E of measure zero and a subset F of E, the measure of F is zero), note that $\mu^*(E) = 0$, so that $\mu^*(F) = 0$ by countable monotonicity. For a fixed $A \subset X$, $\mu^*(A \cap F) = 0$ and $\mu^*(A \cap F^c) \leq \mu^*(A)$ so that $\mu^*(A) \geq \mu^*(A \cap F^c) + \mu^*(A \cap F)$. Thus, F is measurable with respect to μ^*

To summarise, we have a function μ defined on S. From this, we constructed and outer measure μ^* over 2^X and from this, we constructed $\overline{\mu}$ defined on $\mathfrak{M}$.

Do μ^* and μ always agree on S? Not necessarily so, as we are forcing even definition of the empty set.

Think of S as a collection of intervals, with μ measuring length of the interval. It would be really nice if μ^* did the same and even better when $S \subset \mathfrak{M}$ (i.e. $\overline{\mu}|_S = \mu$). When does this happen? The answer to this is given in the Lebesgue-Carathéodory Theorem.

1.3.1 Lebesgue-Carathéodory Theorem

Call S a **semi-ring** if

1. $A, B \in S \Longrightarrow A \cap B \in S$
2. $A, B \in S \Longrightarrow \exists$ a finite disjoint family $\{C_i : 1 \leq i \leq n\}$ in S such that

$$A \backslash B = \bigcup_{i=1}^{n} C_i$$

The last requirement is much weaker than requiring $A \backslash B$ to be in S. Also, note that empty unions are allowed.

Example 25 *Let S be the collection of all intervals of the form $[a, b)$. Half open intervals are needed to ensure the 2nd condition holds. For example, for $[c, d) \subset [a, b)$ with $c < a$ and $d < b$, then $[a, b) \setminus [c, d)$ is another half open interval. However, if we had only open intervals and an interval B was properly contained in another A, then $A \backslash B$ would have been half open so that S would not have been closed under relative complements.*

Problem 26 *Let X and Y be two non-empty sets and let S_X be a semiring on X and S_Y be a semiring on Y. Denote $Z = X \times Y$ and*

$$S_Z = \{A \times B : A \in S_X, B \in S_Y\}$$

Prove that S_Z is a semiring on Z

Solution 27 *Let $A, B \in S_Z$. Then, we can write $A = A_1 \times B_1$ and $B = A_2 \times B_2$, where $A_1, A_2 \in S_X$ and $B_1, B_2 \in S_Y$. We know that*

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

Since S_X and S_Y are semi-rings, therefore $A_1 \cap A_2 \in S_X$ and $B_1 \cap B_2 \in S_Y$. Let $\mathcal{A} = A_1 \cap A_2$ and $\mathcal{B} = B_1 \cap B_2$. Then, $A \cap B = \mathcal{A} \times \mathcal{B}$, where $\mathcal{A} \in S_X$ and $\mathcal{B} \in S_Y$. Thus, $A \cap B \in S_Z$.

Next, let $A, B \in S_Z$ and so, $A = A_1 \times B_1$ and $B = A_2 \times B_2$, where $A_1, A_2 \in S_X$ and $B_1, B_2 \in S_Y \implies \exists$ a finite disjoint family $\{C_i : 1 \le i \le n\} \subset S_X$ and $\{D_i : 1 \le i \le m\} \subset S_Y$ such that

$$A_1 \backslash A_2 = \bigcup_{i=1}^{n} C_i \text{ and } B_1 \backslash B_2 = \bigcup_{i=1}^{m} D_i$$

Now,

$$\begin{aligned} A \backslash B &= (A_1 \times B_1) \setminus (A_2 \times B_2) = ((A_1 \backslash A_2) \times B_1) \cup (A_1 \times (B_1 \backslash B_2)) \\ &= \left(\left(\bigcup_{i=1}^{n} C_i \right) \times B_1 \right) \cup \left(A_1 \times \left(\bigcup_{i=1}^{m} D_i \right) \right) \end{aligned}$$

Note that $A_1 = (A_1 \backslash A_2) \cup (A_1 \cap A_2)$ and $B_1 = (B_1 \backslash B_2) \cup (B_1 \cap B_2)$ so that

$$A \backslash B = \left(\left(\bigcup_{i=1}^{n} C_i \right) \times ((B_1 \backslash B_2) \cup (B_1 \cap B_2)) \right) \cup \left(((A_1 \backslash A_2) \cup (A_1 \cap A_2)) \times \left(\bigcup_{i=1}^{n} D_i \right) \right)$$

The first term becomes

$$\left(\left(\bigcup_{i=1}^{n} C_i \right) \times (B_1 \backslash B_2) \right) \cup \left(\left(\bigcup_{i=1}^{n} C_i \right) \times (B_1 \cap B_2) \right)$$

The second term becomes

$$\left((A_1 \backslash A_2) \times \left(\bigcup_{i=1}^{m} D_i \right) \right) \cup \left((A_1 \cap A_2) \times \left(\bigcup_{i=1}^{m} D_i \right) \right)$$

For the first term, note that $B_1 \backslash B_2$ is disjoint from $B_1 \cap B_2$ so that the first term, which comprises of sets each from S_X and S_Y, is a disjoint union of two sets in S_Z. Similarly, the other term is a disjoint union of two sets in S_Z. All in all,

$$A \backslash B = \left(\left(\bigcup_{i=1}^{n} C_i \right) \times \left(\bigcup_{i=1}^{m} D_i \right) \right) \cup \left(\left(\bigcup_{i=1}^{n} C_i \right) \times (B_1 \cap B_2) \right) \cup \left((A_1 \cap A_2) \times \left(\bigcup_{i=1}^{m} D_i \right) \right)$$

which is a union of disjoint sets, so that the second axiom holds. If we allow for empty unions, then $\varnothing \in S_Z$

If S is a semi-ring, we can define R_S to be the collection of all finite disjoint unions of sets from S. Then, R_S is a ring i.e. a collection of sets closed under relative complements and unions.

Proof. $A \in R \implies A \backslash A \in R_S \implies \varnothing \in R_S$. Let $A, B \in R_S$. Then, $\exists \{A_i : 1 \leq i \leq n\}, \{B_i : 1 \leq i \leq m\} \subset S$ such that

$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{i=1}^{m} B_i$$

and $A_i \cap A_j = \varnothing = B_i \cap B_j$ for $i \neq j$. Then,

$$A \backslash B = A \cap B^c = \bigcup_{i=1}^{n} (A_i \cap B^c) = \bigcup_{i=1}^{n} C_i$$

Note that $C_i \cap C_j = A_i \cap B^c \cap A_j \cap B^c = \varnothing$ so that $A \backslash B \in R_S$

Generally, the union of two collections of disjoint sets may not be disjoint. In order to make sure that $A \cup B$ is in R_S, note that $A \cup B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$ is a disjoint union. Thus, to show that $A \cup B \in R_S$, we need to show that $A \cap B \in R_S$ for $A, B \in R_S$:

$A \cap B = \varnothing \cup (A \cap B)$
$= (A \cap A^c) \cup (A \cap B)$
$= A \cap (A^c \cup B) = A \cap (A \cap B^c)^c$
$= A \backslash (A \backslash B)$, thus $A \cap B \in R_S$ ■

Lemma 28 *Let S be a semiring and $\mu : S \longrightarrow [0, \infty]$ be a pre-measure. Then, μ extends to R_S*

Proof. Let $\overline{\mu} : R_S \longrightarrow [0, \infty]$ be the extension of μ. If $A \in R_S$, then

$$A = \bigcup_{i=1}^{n} A_i$$

for disjoint A_i's in S. Define $\overline{\mu}(A) = \mu(A_1) + ... + \mu(A_n)$. To show that this is well-defined, let

$$A = \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{m} B_i$$

for disjoint A_i's and B_i's in S. Note that

$$\begin{aligned} \bigcup_{i=1}^{n} A_i &= \bigcup_{i=1}^{m} B_i \\ &\implies A_j \cap \bigcup_{i=1}^{n} A_i = A_j \cap \bigcup_{i=1}^{m} B_i \\ &\implies A_j = \bigcup_{i=1}^{m} (A_j \cap B_i). \end{aligned}$$

Then $A_j \cap B_k$ for every k is in S. That is, $\{A_j \cap B_k : 1 \le k \le m\} \subset S$. This collection is also pairwise disjoint. Then, $\mu(A_j) = \mu(A_j \cap B_1) + \mu(A_j \cap B_2) + ... + \mu(A_j \cap B_m)$ so that

$$\sum_{j=1}^{n} \mu(A_j) = \sum_{j=1}^{n}\sum_{k=1}^{m} \mu(A_j \cap B_k).$$

The same argument can be repeated with B replaced with A and we can get the same for

$$\sum_{k=1}^{m} \mu(B_k),$$

except with the summation reversed. Let us do this.

$$\begin{aligned} \bigcup_{i=1}^{n} A_i &= \bigcup_{i=1}^{m} B_i \\ &\implies B_i \cap \bigcup_{i=1}^{n} A_i = B_i \cap \bigcup_{i=1}^{m} B_i \\ &\implies B_i = \bigcup_{i=1}^{n} (B_i \cap A_k) \end{aligned}$$

Then, $A_k \cap B_i$ for every k is in S. That is, $\{A_k \cap B_i : 1 \le k \le n\} \subset S$. This collection is also pairwise disjoint. Then, $\mu(B_i) = \mu(B_i \cap A_1) + \mu(B_i \cap A_2) + ... + \mu(B_i \cap A_n)$ so that

$$\sum_{i=1}^{m} \mu(B_i) = \sum_{i=1}^{m}\sum_{k=1}^{n} \mu(A_k \cap B_i)$$

This and the previous double sum tells us that

$$\sum_{i=1}^{m} \mu(B_i) = \sum_{i=1}^{n} \mu(A_i)$$

so that $\overline{\mu}$ is well-defined.

P1 Since μ is a pre-measure, $\overline{\mu}(\varnothing) = \mu(\varnothing) = 0$.

P2 We need to show that $\overline{\mu}$ is finitely additive. Let $\{A_i : 1 \le i \le n\} \subset R_S$ be a pairwise disjoint collection. Since $A_i \in R_S$, $\exists B_j^{(i)} \in S$ such that

$$A_i = \bigcup_{j=1}^{m_i} B_j^{(i)}$$

with $B_i \cap B_j = \varnothing$ for $i \ne j$. Then,

$$\overline{\mu}\left(\bigcup_{i=1}^{n} A_i\right) = \overline{\mu}\left(\bigcup_{i=1}^{n}\bigcup_{j=1}^{m_i} B_j^{(i)}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \mu\left(B_j^{(i)}\right)$$

since each $B_j^{(i)}$ is disjoint. Thus,

$$\overline{\mu}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \overline{\mu}\left(A_i\right)$$

Before the next property is established, we note that $\overline{\mu}$ is monotone: let $A \subset B$ with $A, B \in R_S$. Then, $\exists A_i, B_j \in S$ such that

$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{m} B_j$$

Now, we have that

$$\overline{\mu}\left(A\right) = \sum_{i=1}^{n} \mu\left(A_i\right) = \mu\left(\bigcup_{i=1}^{n} A_i\right) \text{ and } \overline{\mu}\left(B\right) = \sum_{j=1}^{m} \mu\left(B_j\right) = \mu\left(\bigcup_{j=1}^{m} B_j\right)$$

where the first equalities follow from definition of $\overline{\mu}$ and the other by finite additivity of μ, a pre-measure. By countable monotonicity (hence finite monotonicity of μ) and finite additivity

$$\bigcup_{i=1}^{n} A_i \subset \bigcup_{j=1}^{m} B_j \implies \sum_{i=1}^{n} \mu\left(A_i\right) \leq \sum_{j=1}^{m} \mu\left(B_j\right)$$

so that $\overline{\mu}\left(A\right) \leq \overline{\mu}\left(B\right)$.

P3 Let $A \in R_S$,

$$A \subset \bigcup_{i=1}^{\infty} A_i$$

and $A_i \in R_S$. We need to show that

$$\overline{\mu}\left(A\right) \leq \sum_{k=1}^{\infty} \overline{\mu}\left(A_j\right)$$

For this, define

$$\widetilde{A}_n := A_n \backslash \bigcup_{i=1}^{n-1} A_i \in R_S,\ A := \bigcup_{i=1}^{\infty} \widetilde{A}_i \text{ which gives us } \overline{\mu}\left(\widetilde{A}_n\right) \leq \overline{\mu}\left(A_n\right)$$

Next, note so that $\widetilde{A}_n \in R_S$ so we can let

$$\widetilde{A}_i = \bigcup_{k=1}^{n_i} C_k^{(i)}$$

where $C_k^{(i)}$ are pairwise disjoint and $C_k^{(i)} \in S$ for each k. Then,

$$A \subset \bigcup_{i=1}^{\infty} \widetilde{A}_i \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{n_j} C_k^{(i)}$$

The latter is still countable. Thus, WLOG, we can say that $A_j \in S$ is disjoint.

Since $A \in R_S$, we must have

$$A = \bigcup_{i=1}^{m} B_i$$

where B_i's are from S and are pairwise disjoint. Then

$$\begin{aligned} A &\subset \bigcup_{i=1}^{\infty} A_i \\ &\implies \bigcup_{i=1}^{m} B_i \subset \bigcup_{i=1}^{\infty} A_i \\ &\implies B_k \cap \bigcup_{i=1}^{m} B_i \subset B_k \cap \bigcup_{i=1}^{\infty} A_i \\ &\implies B_k \subset \bigcup_{i=1}^{\infty} (A_i \cap B_k) \\ &\implies \mu(B_k) \leq \sum_{k=1}^{\infty} \mu(A_i \cap B_k). \end{aligned}$$

where the last line follows since μ is a pre-measure. By definition of $\overline{\mu}$,

$$\overline{\mu}(A) = \sum_{k=1}^{m} \mu(B_k) \leq \sum_{k=1}^{m} \sum_{i=1}^{\infty} \mu(A_i \cap B_k) = \sum_{i=1}^{\infty} \sum_{k=1}^{m} \mu(A_i \cap B_k) = \sum_{i=1}^{\infty} \mu(A_i)$$

where the last interchange of summations is possible since every summand is positive and finite. ■

With all this machinery, we can finally answer the question we set before the beginning of this subsection.

Theorem 29 (Lebesgue-Carathéodory) *Let S be a semiring and $\mu : S \longrightarrow [0, \infty]$ be a pre-measure. Then, if we define μ^*, outer measure, $\mathfrak{A}$, a σ-algebra and $\overline{\mu} = \mu^*|_{\mathfrak{M}}$, we will have (a) $S \subset \mathfrak{A}$ and (b) $\overline{\mu}$ and μ agree on S.*

Proof. By **Lemma 28**, instead of the semi-ring S, we can just as well use the ring R_S. Let $A \in R_S$. We want $A \in \mathfrak{A}$. That is, $\mu^*(E) = \mu(A \cap E) + \mu(A^c \cap E)$ for every $E \subset X$. Fix $E \subset X$. By definition of outer measure μ^* (properties of infimum), for any $\epsilon > 0$, we can have

$$E \subset \bigcup_{i=1}^{\infty} E_i \implies \mu^*(E) \geq \sum_{i=1}^{\infty} \mu(E_i) - \epsilon$$

with $E_i \in R_S$. Since $E_i, A \in R_S$, we can have $E_i \backslash A = E_i \cap A^c \in R_S$ and $E_i \cap A \in R_S$. Then, $\mu(E_i) = \mu(E_i \backslash A) + \mu(E_i \cap A)$ so that

$$\begin{aligned}\sum_{i=1}^{\infty} \mu(E_i) &= \sum_{i=1}^{\infty} \mu(E_i \cap A^c) + \sum_{i=1}^{\infty} \mu(E_i \cap A) \\ &\geq \sum_{i=1}^{\infty} \mu(E_i \cap A^c) + \mu^*(E \cap A) \\ &\geq \mu^*(E \backslash A) + \mu^*(E \cap A)\end{aligned}$$

That is,

$$\mu^*(E) + \epsilon \geq \sum_{i=1}^{\infty} \mu(E_j) \geq \mu^*(E \backslash A) + \mu^*(E \cap A)$$

Since this inequality holds for any ϵ, we can let it go to zero and we will then be done (the other inequality holds, as well). Thus, A is μ^*-measurable. That is, $A \in \mathfrak{A}$.

(b) If $\mu(A) = \infty$, then $\overline{\mu}(A) = \mu(A)$, trivially. Assume otherwise. We divide the proof into two steps: for any $A \in \mathfrak{A}$, $\overline{\mu}(A) \leq \mu(A)$ and, conversely, $\mu(A) \leq \overline{\mu}(A)$. The first is immediate: to recall,

$$\mu^*(E) = \inf \sum_{i=1}^{\infty} \mu(E_i)$$

where inf is taken over $\{E_i : i \in \mathbb{N}\} \subset S$ such that this family covers E. Since $A \subset A$ is a covering of itself and since $\mu^*(A) = \overline{\mu}(A)$ for $A \in \mathfrak{A}$, then $\overline{\mu}(A) \leq \mu(A)$. For the second inequality, let

$$A \subset \bigcup_{i=1}^{\infty} A_i$$

with $A_i \in S$ for each i and let

$$B_i = A_i \backslash \bigcup_{n=1}^{i-1} A_n$$

Then,

$$\begin{aligned}B_i \cap B_j &= \left(A_i \backslash \bigcup_{k=1}^{i-1} A_k\right) \cap \left(A_j \backslash \bigcup_{m=1}^{j-1} A_m\right) \\ &= A_i \cap \bigcap_{k=1}^{i-1} A_k^c \cap A_j \cap \bigcap_{m=1}^{j-1} A_m^c = \varnothing\end{aligned}$$

since the index $A_i \cap A_{m_0}^c = \varnothing$ for some $1 \leq m_0 < j$. Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

since $B_i \subset A_i$ for each i. We will be done if we can prove that

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

Since $B_i \subset A_i$ for each i, we must have

$$\bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} A_i$$

For the converse, let

$$x \in \bigcup_{i=1}^{\infty} A_i$$

Define $I = \{j : x \in A_j\} \subset \mathbb{N}$. By the well-ordering principle, I has a least element, say m. Then,

$$x \in B_m \implies x \in \bigcup_{i=1}^{\infty} B_i.$$

so that both sides of the inclusion hold. ■

For example, consider the collection $S = \{[a, b) : a, b \in \mathbb{R}\}$ for subsets of $\mathbb{R}$ and define $\mu([a, b)) = b - a$. Then, we can have complete measure $\overline{\mu}$ defined on some σ-algebra $\mathfrak{M}$ and $\mathfrak{m}_1([a, b)) = \overline{\mu}([a, b)) = b - a$ and $\mathfrak{m}^*$ is the outer measure, the intermediate step. Such a measure is given a special name, called the Lebesgue Measure and is covered in the next section.

1.4 Lebesgue Measure

Let $X = \mathbb{R}^p$ for some integer p. Define a half-open rectangle $R = [a_1, b_1) \times ... \times [a_p, b_p)$. Such a collection of R's forms a semi-ring, as already been discussed for $p = 1$. The pre-measure μ on such a semi-ring may be defined as the volume of each half-open rectangle $\mu : S \longrightarrow [0, \infty]$ by $\mu(R) = (b_1 - a_1) ... (b_p - a_p)$. It is easy to see that $\mu(\varnothing) = 0$ and that μ is finitely additive. Countable monotonicity requires some work. Assume $p = 1$. Our goal will be to extract a finite cover.

Let

$$[a, b) \subset \bigcup_{i=1}^{\infty} [a_i, b_i)$$

and let $\epsilon > 0$. There exists a j such that $b - \epsilon = b_j$. Define $\widetilde{b}_j = b_j + \epsilon$ for only this j and leave the others fixed. Similarly, there exists an i such that $a + \epsilon > a_i$. Define $\widetilde{a}_i = a_i - \epsilon$. Then, the collection

$$\left\{\left(\widetilde{a}_k, \widetilde{b}_k\right) : k \in \mathbb{N}\right\}$$

covers $[a, b]$ so that we can have a finite subcover

$$\left\{\left(\widetilde{a}_k, \widetilde{b}_k\right) : 1 \leq i \leq N\right\}$$

and so

$$b-a \leq \sum_{k=1}^{N} \widetilde{b}_k - \widetilde{a}_k = \sum_{k=1}^{N} b_k - a_k + 2\epsilon \leq \sum_{k=1}^{\infty} b_k - a_k + 2\epsilon$$

Since this is true for every ϵ, we can let it tend to zero. Thus, from this pre-measure on the semiring S, we can construct an outer measure on $2^{\mathbb{R}}$ and use that to form a measure on a σ-algebra $\mathfrak{M}$.

This argument can be generalised.

Problem 30 *Let $f : I \longrightarrow \mathbb{R}$ be continuous for any interval $I \subset \mathbb{R}$. Prove that if f is increasing and continuous, then $\nu([a,b)) := f(b) - f(a)$ is a pre-measure on the semiring S of half-open intervals $[a,b)$*

Solution 31 *Let $a, b \in \mathbb{R}$ and $a \leq b$. Then, $f(a) \leq f(b)$ so that $\nu([a,b)) \geq 0$. Hence the function is well-defined. Also, since $[a,a) = \varnothing$ is an interval for any $a \in \mathbb{R}$, then $\nu([a,a)) = f(a) - f(a) = 0$. Thus, $\nu(\varnothing) = 0$.*

Now, let

$$E = [a,b) = \bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} [\alpha_i, \beta_i)$$

where $\{E_i : i \in \mathbb{N}\} \subset S$ and $E_i \cap E_j = \varnothing$. If the E_i's are already ordered, then, since the union of two disjoint intervals I_1 and I_2 is an interval if $\inf I_2 = \sup I_2$ we can have $a = \alpha_1 \leq \beta_1 = \alpha_2 \leq \beta_2 = \alpha_3 \leq ... = \alpha_n \leq \beta_n = b$. If E_i's are not ordered, then for $k = 1, ..., n$, let $a_k = \min_{i=k}^{n} \alpha_i$, $b_k = \min_{i=k}^{n} \beta_i$. It follows that we have the chain $a = a_1 \leq b_1 = a_2 \leq b_2 = a_3 \leq ... = a_n \leq b_n = b$ and

$$\begin{aligned}
\nu(E) &= f(b) - f(a) = f(b_n) + \sum_{i=2}^{n} f(a_i) - \sum_{i=2}^{n} f(a_i) - f(a_1) \\
&= f(b_n) + \sum_{i=2}^{n} f(b_i) - \sum_{i=2}^{n} f(a_i) - f(a_1) \\
&= \sum_{i=1}^{n} f(b_i) - \sum_{i=1}^{n} f(a_i) = \sum_{i=1}^{n} (f(b_i) - f(a_i)) = \sum_{i=1}^{n} \nu(E_i)
\end{aligned}$$

Thus, ν is finitely additive. We prove that ν is monotone: let $[a,b) \subset [c,d)$ with $c \leq a < b \leq d$. Then, by monotonicity of f, $f(b) - f(a) \leq f(d) - f(a) \leq f(d) - f(c)$. Thus, $\nu([a,b)) \leq \nu([c,d))$.

These two combined tell us that ν is finitely monotone. We now prove that ν is countably monotone.

Let

$$E = [a,b) \subset \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} [a_i, b_i)$$

Let $\epsilon > 0$ so that, by continuity of f, for every $\epsilon > 0$, we can have we can have $\delta_i > 0$ and $\delta > 0$ such that $a_i < \delta_i + a_i \Longrightarrow f(a_i) - f(a_i - \delta_i) < \frac{\epsilon}{2^i}$ and $b - \delta < b \Longrightarrow f(b) - f(b - \delta) < \frac{\epsilon}{2}$

Now,

$$[a, b-\delta] \subset [a, b) \subset \bigcup_{i=1}^{\infty} [a_i, b_i) \subset \bigcup_{i=1}^{\infty} (a_i - \delta_i, b_i)$$

That is,

$$[a, b-\delta] \subset \bigcup_{i=1}^{\infty} (a_i - \delta_i, b_i)$$

Since $[a, b-\delta]$ *is compact and*

$$\bigcup_{i=1}^{\infty} (a_i - \delta_i, b_i)$$

is an open cover, we can extract a finite subcover, say

$$[a, b-\delta] \subset \bigcup_{i=1}^{n} (a_i - \delta_i, b_i)$$

By properties of ν *already established (monotonicity and, therefore, finite sub-additivity), we have*

$$f(b-\delta) - f(a) \leq \sum_{i=1}^{n} f(b_i) - f(a_i - \delta_i)$$

Now,

$$\begin{aligned} f(b) - f(a) &< f(b-\delta) - f(a) + \frac{\epsilon}{2} \\ &\leq \sum_{i=1}^{n} \left(f(b_i) - f(a_i - \delta_i)\right) + \frac{\epsilon}{2} \\ &< \sum_{i=1}^{n} \left(f(b_i) - f(a_i) + \frac{\epsilon}{2^{i+1}}\right) + \frac{\epsilon}{2} \\ &= \sum_{i=1}^{n} f(b_i) - f(a_i) + \sum_{i=1}^{n} \frac{\epsilon}{2^{i+1}} + \frac{\epsilon}{2} \end{aligned}$$

Letting $n \to \infty$ *gives us*

$$f(b) - f(a) \leq \sum_{i=1}^{\infty} f(b_i) - f(a_i) + \epsilon$$

We can now let $\epsilon \to 0$ *to get the required countable monotonicity.*

For $f(t) = t$, this defines the Lebesgue measure.

As of now, the pre-measure μ defined in the beginning of this section is undefined on single points. However, we can extend this μ to the Lebesgue measure $\mathfrak{m}_p$ on the σ-algebra $\mathfrak{M}_p$, in which case $\mathfrak{m}_p(\{a\}) = 0$ and $\mathfrak{m}_p(A) = 0$ for any countable A. The measure $\mathfrak{m}_p$ is also translation invariant: $\mathfrak{m}_p(a + A) = \mathfrak{m}_p(A)$ if $A \in \mathfrak{M}_p$. In particular, $\mathfrak{m}_1(\mathbb{Q}) = 0$ and, therefore, $\mathfrak{m}_1(\mathbb{Q}^c \cap [0,1]) = 1$. This example shows that everywhere dense sets have zero measure.

Problem 32 *Let X be any non-empty set, $S \subset 2^X$ and $\mu : S \longrightarrow [0,\infty]$ be a function such that there exists a σ-algebra $\mathfrak{A}$ and a measure $\overline{\mu}$ on $\mathfrak{A}$ such that $S \subset \mathfrak{A}$ and for any $A \in S$, we have $\mu(A) = \overline{\mu}(A)$. Is it true that μ is a pre-measure?*

Solution 33 *No. We are not guaranteed that the empty set is in S, so we cannot even say that $\mu(\varnothing) = 0$. It is true that μ is finite additive and countably monotone by definition (since every measure is) but the very first requirement for a pre-measure may fail to hold.*

Definition 34 *Let (X,τ) be a topological space. We call G a G_δ set if*

$$G = \bigcap_{i=1}^{\infty} O_i$$

where O_i is open for each i. We call E a F_σ-set if

$$E = \bigcup_{i=1}^{\infty} F_i$$

where F_i is closed for each i.

These sets are complements of each other.

Theorem 35 *The following are equivalent*

1. *$E \in \mathfrak{M}_p$*
2. *$\forall \epsilon > 0$, there exists an open O with $E \subset O$ and $\mathfrak{m}_p(O \backslash E) < \epsilon$*
3. *There exists a G_δ set G with $E \subset G$ and $\mathfrak{m}_p(G \backslash E) = 0$*
4. *$\forall \epsilon > 0$, there exists a closed set F with $F \subset E$ and $\mathfrak{m}_p(E \backslash F) < \epsilon$*
5. *There exists an F_σ set F with $F \subset E$ and $\mathfrak{m}_p(E \backslash F) = 0$*

Proof. ($1 \implies 2$)

Let $E \in \mathfrak{M}_p$ with $\mathfrak{m}_p(E) < \infty$. Then, by Lebesgue-Carathéodory Theorem, $\mathfrak{m}_p(E) = \mathfrak{m}_p^*(E)$ so that

$$\mathfrak{m}_p(E) = \inf \sum_{i=1}^{\infty} \text{Volume}(R_i)$$

By the properties of infimum, we can choose a family $\{R_i : i \in \mathbb{N}\}$ with

$$\mathfrak{m}_p^*(E) > \sum_{i=1}^{\infty} \text{Volume}(R_i) - \frac{\epsilon}{2}$$

Enlarge each R_i to an open $\widetilde{R}_i$ so that

$$\text{Volume}\left(\widetilde{R}_i\right) = \text{Volume}(R_i) + \frac{\epsilon}{2^{i+1}}$$

Then,

$$O = \bigcup_{i=1}^{\infty} \widetilde{R}_i$$

is a countable union of open sets and is, therefore, open with

$$\mathfrak{m}_p(O) \le \sum_{i=1}^{\infty} \mathfrak{m}_p\left(\widetilde{R}_i\right) = \sum_{i=1}^{\infty}\left(\mathfrak{m}_p(R_i) + \frac{\epsilon}{2^{i+1}}\right) = \frac{\epsilon}{2} + \sum_{i=1}^{\infty} \mathfrak{m}_p(R_i) < \mathfrak{m}_p(E) + \epsilon$$

That is, $\mathfrak{m}_p(O \backslash E) = \mathfrak{m}_p(O) - \mathfrak{m}_p(E) < \epsilon$

If $\mathfrak{m}_p(E) = \infty$, we can take $R_j = \left[-2^j, 2^j\right) \times ... \times \left[-2^j, 2^j\right)$ and let $E_j = E \cap R_j$ so that $\mathfrak{m}_p(E_j) < \infty$. Now take open sets O_j with $E_j \subset O_j$ such that $\mathfrak{m}_p(O_j \backslash E_j) < \frac{\epsilon}{2^j}$. With this, we can have an open set

$$O = \bigcup_{i=1}^{\infty} O_i \text{ with } E \subset O \implies O \backslash E \subset \bigcup_{i=1}^{\infty}(O_i \backslash E_i) \implies \mathfrak{m}_p(O \backslash E) < \sum_{i=1}^{\infty} \mathfrak{m}_p(O_i \backslash E_i) < \epsilon$$

$(2 \implies 3)$

For $\epsilon = \frac{1}{k}$, we can have $E \subset O_k$ for each k so that

$$G = \bigcap_{i=1}^{\infty} O_k \supset E \implies \mathfrak{m}_p(G \backslash E) \le \mathfrak{m}_p(O_k \backslash E) < \frac{1}{k} \implies \mathfrak{m}_p(G \backslash E) = 0$$

$(3 \implies 1)$

We know that there is a G_δ set G with $E \subset G$ and $\mathfrak{m}_p^*(G \backslash E) = 0$ (note the use of star here!). Thus, $G \backslash E$ is measurable with respect to $\mathfrak{m}_p^*$ by **Proposition 20** and so, $G \backslash E \in \mathfrak{M}_p$. Since G is the union of open sets, clearly $G \in \mathfrak{M}_p$. Since $E \subset G$, we have $E = G \backslash (G \backslash E)$. Thus, E is measurable.

$(1 \implies 4)$

Let $E \in \mathfrak{M}_p$ and let $\epsilon > 0$. Since $\mathfrak{M}_p$ is a σ-algebra, we must have $E^c \in \mathfrak{M}_p$. Then, there is an open set O with $E^c \subset O$ and $\mathfrak{m}_p(O \backslash E^c) < \epsilon$. Now, $E^c \subset O \implies O^c \subset E^{cc} = E$. That is, $\mathfrak{m}_p(O \backslash E^c) = \mathfrak{m}_p(O) - \mathfrak{m}_p(E^c)$. Now, note that $\mathfrak{m}_p(X) = \mathfrak{m}_p(O) + \mathfrak{m}_p(O^c)$ and, similarly, $\mathfrak{m}_p(X) = \mathfrak{m}_p(E) + \mathfrak{m}_p(E^c)$. We, therefore have $\mathfrak{m}_p(X) - \mathfrak{m}_p(X) = \mathfrak{m}_p(O) + \mathfrak{m}_p(O^c) - \mathfrak{m}_p(E) - \mathfrak{m}_p(E^c) = 0$ so that $\mathfrak{m}_p(O \backslash E^c) = \mathfrak{m}_p(O) - \mathfrak{m}_p(E^c) = \mathfrak{m}_p(E) - \mathfrak{m}_p(O^c) = \mathfrak{m}_p(E \backslash O^c)$. Since O^c is closed, by (2), the proof is done.

$(4 \implies 5)$

Let $\epsilon = \frac{1}{n}$. Then, there exists a closed set F_n with $F_n \subset E$ and $\mathfrak{m}_p (E \backslash F_n) < \frac{1}{n}$. By definition,

$$F = \bigcup_{n=1}^{\infty} F_n$$

tells us that F is a F_σ-set. Also, since $F_n \subset F$ so that $E \backslash F \subset E \backslash F_n$ and so, by monotonicity of $\mathfrak{m}_p$, $\mathfrak{m}_p (E \backslash F) \leq \mathfrak{m}_p (E \backslash F_n) < \frac{1}{n}$. Letting $n \to \infty$ gives us $\mathfrak{m}_p (E \backslash F) \leq 0$. That is, $\mathfrak{m}_p (E \backslash F) = 0$.

$(5 \implies 1)$

We know that there is a F_σ set F with $F \subset E$ and $\mathfrak{m}_p^* (E \backslash F) = 0$. That is, $E \backslash F$ is $\mathfrak{m}_p^*$-measurable and so $E \backslash F \in \mathfrak{M}_p$. Moreover, $F \in \mathfrak{M}_p$ because it is the intersection of open sets. Since $F \subset E$, we have $E = E \cap X = E \cap (F \cup F^c) = (F \cup E) \cap (F^c \cup E) = F \cup (E \cap F^c) = F \cup (E \backslash F)$. Since $E \backslash F, F \in \mathfrak{M}_p$, it follows that $E = F \cup (E \backslash F) \in \mathfrak{M}_p$, since any σ-algebra is closed under unions. Thus, E is measurable. ■

This tells us something very important:

Theorem 36 (Regularity of Lebesgue Measure) *Assume that* $E \in \mathfrak{M}_p$. *Then,* $\mathfrak{m}_p (E) \overset{(1)}{=} \inf \{\mathfrak{m}_p (O) : O \textit{ is open and } E \subset O\}$
$\overset{(2)}{=} \sup \{\mathfrak{m}_p (K) : K \textit{ is compact and } K \subset E\}$

Proof. By properties of infimum and (2) in **Theorem 35**, The first equality follows directly. For the second equality, by (4) in **Theorem 35**, we need this only for closed sets E. Let $R_j = \left[-2^j, 2^j\right] \times ... \times \left[-2^j, 2^j\right]$ and let $K_j = E \cap R_j$. Then, K_j is compact. Note that $K_1 \subset K_2 \subset ....$ Then,

$$E = \bigcup_{i=1}^{\infty} K_i \implies \mathfrak{m}_p (E) = \lim_{n \to \infty} \mathfrak{m}_p (K_n) = \sup \mathfrak{m}_p (K_n)$$

which establishes the second equality. ■

1.4.1 A non-measurable Set

And now, we will construct a non-measurable set $E \subset [0,1]$ such that $E \in \mathfrak{M}_p$. Call $x, y \in [0,1]$ equivalent if $x - y \in \mathbb{Q}$. Thus, this relation partitions $[0,1]$. That is,

$$[0,1] = \bigcup_{\alpha} A_\alpha$$

where α is a representative of each equivalence class. That is, if $x, y \in A_\alpha$, then $x - y \in \mathbb{Q}$. In this indexing, the Axiom of Choice is needed (there exists a set $C \subset [0,1]$ such that for every α, C has exactly one element from A_α). This C is not measurable. Assume that it is.

If $q, r \in \mathbb{Q}$ with $q \neq r$, then $C + q$ and $C + r$ are disjoint because if $z \in (C + q) \cap (C + r)$, then $z = x + q$ and $z = y + r$ for some $x, y \in C$. Then, $x \sim y$ but that would mean $x = y$ so that $q = r$, a contradiction.

Thus, $C+q$ and $C+r$ are two distinct measurable sets. That is, $C+q \in \mathfrak{M}_p$ for every q. It is easy to prove that

$$\bigcup_{q\in\mathbb{Q}\cap[0,1]} (C+q) \subset [0,2]$$

which is a countable union of disjoint sets. Thus,

$$\begin{aligned} 2 &= \mathfrak{m}_p([0,2]) \geq \mathfrak{m}_p\left(\bigcup_{q\in\mathbb{Q}}(C+q)\right) \\ &= \sum_{q\in\mathbb{Q}} \mathfrak{m}_p(C+q) = \sum_{q\in\mathbb{Q}} \mathfrak{m}_p(C) \end{aligned}$$

which is either 0 or infinity. The latter is a contradiction so that $\mathfrak{m}_p(C) = 0$. However,

$$[0,1] \subset \bigcup_{q\in\mathbb{Q}} (C+q) \implies 1 = \mathfrak{m}_p([0,1]) \leq \sum_{q\in\mathbb{Q}} \mathfrak{m}_p(C) = 0$$

another contradiction.

Thus, C is not measurable.

Had it been $\mathfrak{m}_p^*$ instead of $\mathfrak{m}_p$, we would not have been able to use countable additivity of measure.

In fact, every set of positive measure contains a non-measurable set.

Proof. As above, let $E \subset \mathbb{R}$ be a measurable subset of $E \in \mathfrak{M}_1$. Call $x, y \in E$ equivalent if $x - y \in \mathbb{Q}$. Let C be the set of representatives for this equivalence relation. C is not measurable. Assume that it is.

As above, $C+q$ and $C+r$ for $q, r \in \mathbb{Q}$ are two distinct measurable sets. That is, $C+q \in \mathfrak{M}_1$ for every q. Consider $C + q \bmod 1$ for every $q \in \mathbb{Q}\cap[0,1)$. Then,

$$\bigcup_{q\in\mathbb{Q}\cap(0,1]} (C + q \bmod 1) = [0,1]$$

so that

$$\mathfrak{m}_1([0,1]) = \sum_{q\in\mathbb{Q}\cap(0,1]} \mathfrak{m}_1(C + q \bmod 1) = \sum_{i=1}^{\infty} \mathfrak{m}_1(C)$$

If C were measurable, then $\mathfrak{m}_1(C) \geq 0$. However, the infinite sum of repeated non-negative numbers is either 0 or infinity, neither of which is equal to $1 = \mathfrak{m}_1([0,1])$. ■

Moreover, the Axiom of Choice is, in fact, equivalent to existence of non-measurable sets.

All in all, this says that $\mathfrak{M}_p \neq 2^{\mathbb{R}^p}$

1.4.2 Cantor Set

There is a unique creature called the Cantor set. It is a set with zero measure and is yet uncountable. It is constructed via an iterative process in which at the n-th step, we delete intervals C_n. The countable intersection of such sets C_n is the Cantor Set C. Usually, the C_n's are formed by deleting the middle third interval at each n-th step but this is allowed to vary. For example, if we consider deleting middle thirds, then $C_1 = [0, 1/3] \cup [2/3, 1]$ and $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ and so on. The Cantor set $C \neq \varnothing$ because $0 \in C$. Moreover, C is measurable since we are taking countable intersection of measurable sets. For each n, it is obvious that $\mathfrak{m}_1(C_n) = 2^n/3^n$ (see **Problem 37**) where the denominator is length of one interval in C_n and the numerator follows from the number of intervals. Moreover, $C \subset C_n$. As $n \to \infty$, by continuity of Lebesgue measure, $\mathfrak{m}_1(C) = 0$.

Problem 37 *C is a perfect.*

Recall that a set is said to be **perfect** if it has no isolated points. Thus, we have to prove that $\forall x \in C$, any neighborhood of x contains another point from C.

Solution 38 *Let $\epsilon > 0$. For $x \in C$, define $N_\epsilon(x) = \{y : |x - y| < \epsilon\}$. We need to show $C \cap N_\epsilon(x) \setminus \{x\} \neq \varnothing$.*

Since $\epsilon > 0$, by the Archimedean property of real numbers, $\exists N$ such $1/3^N < \epsilon$. Since $C_1 = [0, 1/3] \cup [2/3, 1]$. $C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9]$ and, in general, for the N-th step, we have

$$C_N = \bigcup_{k=0}^{3^{N-1}-1} \left(\left[\frac{3k}{3^N}, \frac{3k+1}{3^N} \right] \cup \left[\frac{3k+2}{3^N}, \frac{3k+3}{3^N} \right] \right)$$

and C is formed by the countably infinite intersection of such C_i's. It follows that $x \in \left[\frac{3k}{3^N}, \frac{3k+1}{3^N}\right] = E_k$ (say) or $x \in \left[\frac{3k+2}{3^N}, \frac{3k+3}{3^N}\right] = E_{k'}$ (say). Moreover

$$\mathfrak{m}_1(E_k) = \frac{3k+1}{3^N} - \frac{3k}{3^N} = \frac{1}{3^N} = \frac{3k+3}{3^N} - \frac{3k+2}{3^N} = \mathfrak{m}_1(E_{k'}) < \epsilon$$

Thus, if $x \in E_k$, then $E_k \subset N_\epsilon(x)$. If $x \in E_{k'}$, then $E_{k'} \subset N_\epsilon(x)$.

Now in both cases, in the $N+1$ step, E_k is split into

$$\begin{aligned} & \left[\frac{3^2 k}{3^{N+1}}, \frac{3^2 k + 1}{3^{N+1}} \right] \cup \left[\frac{3^2 k + 2}{3^{N+1}}, \frac{3^2 k + 3}{3^{N+1}} \right] \\ = & \left[\frac{3j}{3^{N+1}}, \frac{3j+1}{3^{N+1}} \right] \cup \left[\frac{3j+2}{3^{N+1}}, \frac{3j+3}{3^{N+1}} \right] = E_k' \cup E_k'' \text{ (say)} \end{aligned}$$

and $\left[\frac{3k+2}{3^N}, \frac{3k+3}{3^N}\right]$ is split into

$$\begin{aligned} & \left[\frac{3^2 k + 6}{3^{N+1}}, \frac{3^2 k + 7}{3^{N+1}} \right] \cup \left[\frac{3^2 k + 8}{3^{N+1}}, \frac{3^2 k + 9}{3^{N+1}} \right] \\ = & \left[\frac{3(j+2)}{3^{N+1}}, \frac{3(j+2)+1}{3^{N+1}} \right] \cup \left[\frac{3(j+2)+8}{3^{N+1}}, \frac{3(j+2)+3}{3^{N+1}} \right] = E_{k'}' \cup E_{k'}'' \text{ (say),} \end{aligned}$$

for $j = 3k$. That is, replacing the dummy variable, we can re-write C_{N+1} as

$$C_{N+1} = \bigcup_{k=0}^{3^N-1} \left(\left[\frac{3k}{3^{N+1}}, \frac{3k+1}{3^{N+1}} \right] \cup \left[\frac{3k+2}{3^{N+1}}, \frac{3k+3}{3^{N+1}} \right] \right)$$

Again, note the

$$\mathfrak{m}_1 \left(E'_{k'} \right) = \frac{1}{3^{N+1}} = \mathfrak{m}_1 \left(E''_{k'} \right) = \mathfrak{m}_1 \left(E'_k \right) = \mathfrak{m}_1 \left(E''_k \right)$$

If $x \in E_k$, then either $x \in E'_k$ or $x \in E''_k$. If $x \in E'_k$, then since $C \subset E''_k \subset E_k \subset N_\epsilon(x)$, we must have $C \cap N_\epsilon(x) \setminus \{x\} \neq \varnothing$. If $x \in E''_k$, then since $C \subset E'_k \subset E_k \subset N_\epsilon(x)$, we must have $C \cap N_\epsilon(x) \setminus \{x\} \neq \varnothing$. Similarly, $x \in E_{k'}$, then either $x \in E'_{k'}$ or $x \in E''_{k'}$. If $x \in E'_{k'}$, then since $C \subset E''_{k'} \subset E_k \subset N_\epsilon(x)$, we must have $C \cap N_\epsilon(x) \setminus \{x\} \neq \varnothing$. If $x \in E''_{k'}$, then since $C \subset E'_{k'} \subset E_2 \subset N_\epsilon(x)$, we must have $C \cap N_\epsilon(x) \setminus \{x\} \neq \varnothing$. In either case, we have that, for any $x \in C$ and any $\epsilon > 0$, $C \cap N_\epsilon(x) \setminus \{x\} \neq \varnothing$.

Theorem 39 *C is uncountable*

Proof. Assume otherwise. Then, we can let $C = \{c_k : k \in \mathbb{N}\}$. C_1 is the union of two disjoint intervals. One of these does not contain c_1. Let $C \cap C_1 = F_1$ be the set which does not contain c_1. F_1 is a union of two parts, one of which does not contain c_2. Call this part (which does not contain c_2) F_2 and so on. Eventually, we end up with a sequence $F_{n+1} \subset F_n \subset C$ where $c_k \notin F_k$. If we have countably many compact sets and all finite intersections are non-empty, then the total intersection is non-empty.

Since we have a decreasing sequence of sets, the intersection

$$F = \bigcap_{n=1}^{\infty} F_n \neq \varnothing$$

Moreover, $F \subset C$. That is, if $x \in F$, then $x \in C$. Hence $x = c_N$ for some N, a contradiction, since $c_N \notin F_N$ and $F \subset F_N \subset C$. ∎

Recall that the Borel σ-algebra $\mathfrak{L}$ is the minimal σ-algebra that contains all open subsets of $\mathbb{R}$, under the usual topology. Since $\mathfrak{L}$ is the smallest σ-algebra on $\mathbb{R}$, we can claim that $\mathfrak{L} \subset \mathfrak{M}_1$. However, the converse does not hold. By completeness of $\mathfrak{M}_1$, any subset of the Cantor set is in $\mathfrak{M}_1$. It can shown that there is such a subset not in $\mathfrak{L}$. Moreover, the completion of $\mathfrak{L}$ gives $\mathfrak{M}_1$ but the proof of this fact will have to wait for now, until we get into a discussion of functions acting on Measure Spaces.

Problem 40 *C is totally disconnected set.*

Solution 41 *We need to show that for every $x \in C$, any neighborhood of x contains a point from the complement of C. That is, we need to show that $\forall x \in C$ and any $\epsilon > 0$, $C^c \cap N_\epsilon(x) \neq \varnothing$. To begin the proof, we observe that*

$$C^c = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right) = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} I_k$$

in $[0,1]$*. Again, since* $\epsilon > 0$*, by the Archimedean property of real numbers,* $\exists N$ *such that* $1/3^{N+1} < \epsilon$*. Note that, at the* N*-th step,* $\mathfrak{m}_1(I_k) = 3^{-N} < 3\mathfrak{m}_1(I_k) = 3^{-N+1} < \epsilon$*. Moreover,* $x \in E_k$ *or* $x \in E_{k'}$*. In either case,* $I_k \cap N_\epsilon(x) \neq \varnothing$ *since* $\mathfrak{m}_1(N_\epsilon(x)) = 2\epsilon > \epsilon > \mathfrak{m}_1(I_k)$ *and* I_k *is in between* E_k *and* $E_{k'}$*. That is,* $C^c \cap N_\epsilon(x) \neq \varnothing$*.*

Problem 42 *Suppose that in the construction of a Cantor set, on the* n*-th step, instead of throwing away the middle third of every interval, we throw away middle intervals of length* $a_n > 0$ *(*a_n *depends only on* n*). Show that, by choosing the right sequence* $\{a_n : n \in \mathbb{N}\}$*, we can make the measure of the resulting set to be any number between 0 and 1. Deduce that there exists an open subset of* $[0,1]$ *whose boundary has positive measure.*

Solution 43 *Let* $x \in [0,1]$*. We want to construct a Cantor set of Lebesgue measure* x*. Since* $x \in [0,1]$*,* $\exists$ *a sequence* $\{b_n : n \in \mathbb{N}\}$ *such that*

$$x = \sum_{n=1}^{\infty} b_n$$

Let

$$1 - x = \sum_{n=1}^{\infty} a_n$$

where a_n *is some sequence dependent on the sequence* $\{b_n : n \in \mathbb{N}\}$*, clearly convergent. Now, begin with* $[0,1]$ *and throw away*

$$\left(\frac{1-a_1}{2}, \frac{1+a_1}{2}\right)$$

which has Lebesgue measure a_1*. The resulting* C_1 *is*

$$C_1 = \left[0, \frac{1-a_1}{2}\right] \cup \left[\frac{1+a_1}{2}, 1\right].$$

The middle point of

$$\left[0, \frac{1-a_1}{2}\right]$$

is

$$\frac{1-a_1}{4}$$

so for the left interval, we throw away

$$\left(\frac{1-a_1-a_2}{4}, \frac{1-a_1+a_2}{4}\right).$$

This set has Lebesgue measure $a_2/2$*. Similarly, the middle point for the right interval is* $\frac{3+a_1}{4}$*, we throw away*

$$\left(\frac{3+a_1}{4} - \frac{a_2}{4}, \frac{3+a_1}{4} + \frac{a_2}{4}\right).$$

This set has Lebesgue measure $a_2/2$, so in total, we have thrown away intervals of length a_2. We are left with

$$\begin{aligned} C_2 &= \left[0, \frac{1-a_1-a_2}{4}\right] \cup \left[\frac{1-a_1+a_2}{4}, \frac{2-2a_1}{4}\right] \\ &\quad \cup \left[\frac{2+2a_1}{4}, \frac{3+a_1-a_2}{4}\right] \cup \left[\frac{3+a_1+a_2}{4}, 1\right]. \end{aligned}$$

For the first set in C_2, we can throw away $\left(\frac{1-a_1-a_2-a_3}{8}, \frac{1-a_1-a_2+a_3}{8}\right)$. For the second, throw away $\left(\frac{a_2+3-3a_1-a_3}{8}, \frac{a_2+3-3a_1+a_3}{8}\right)$; for the third, $\left(\frac{5+3a_1-a_2-a_3}{8}, \frac{5+3a_1-a_2+a_3}{8}\right)$; and for the fourth, $\left(\frac{3+a_1+a_2+4-a_3}{8}, \frac{3+a_1+a_2+4+a_3}{8}\right)$. This gives us

$$\begin{aligned} C_3 = &\left[0, \frac{1-a_1-a_2-a_3}{8}\right] \cup \left[\frac{1-a_1-a_2+a_3}{8}, \frac{2-2a_1-2a_2}{8}\right] \cup \\ &\left[\frac{1-a_1+a_2}{4}, \frac{3-3a_1+a_2-a_3}{8}\right] \cup \left[\frac{3-3a_1+a_2+a_3}{8}, \frac{1-a_1}{2}\right] \cup \\ &\left[\frac{2+2a_1}{4}, \frac{5+3a_1-a_2-a_3}{8}\right] \cup \left[\frac{5+3a_1-a_2+a_3}{8}, \frac{3+a_1-a_2}{4}\right] \cup \\ &\left[\frac{3+a_1+a_2}{4}, \frac{3+a_1+a_2+4-a_3}{8}\right] \cup \left[\frac{3+a_1+a_2+4+a_3}{8}, 1\right] \end{aligned}$$

Continuing this way, we observe that we can throw away intervals of total length a_n at the n-th step. Since $[0,1]$ is the disjoint union of all these open sets and the intersections of C_n, we get that

$$1 = \mathfrak{m}_1(C) + \sum_{n=1}^{\infty} a_n$$

or that $\mathfrak{m}_1(C) = x$.

In this construction, we note that C is countable intersection of finite union of closed sets C_n. Thus, C is closed so that C^c is open. Since $[0,1] = C \cup C^c$ and $\partial C^c = Cl(C^c) - Int(C^c) = Cl(C^c) - C^c = Cl(C^c) \cap C = (C \setminus \{0,1\}) \cap C = C \setminus \{0,1\}$. Thus, $\mathfrak{m}_1(\partial C^c) = \mathfrak{m}_1(C) - \mathfrak{m}_1(\{0,1\}) = \mathfrak{m}_1(C)$. The crucial point here is realizing that $Cl(C^c) = C \setminus \{0,1\}$ and this is because for each n, $\partial C_n \setminus \{0,1\} = Cl(I_n)$ where I_n is the disjoint union of the open sets $I_k^{(n)}$ for which $\mathfrak{m}_1(I_n) = a_n$.

2 Functions on Measure Spaces

2.1 Measurable Functions

Let $(X, \mathfrak{A}_X, \mu_X), (Y, \mathfrak{A}_Y, \mu_Y)$ be a measure space and let $f : X \longrightarrow Y$ be a function. f is said to be a **measurable function** (with respect to $\mathfrak{A}$) if the pre-image of any μ_Y-measurable set is μ_X-measurable. Therefore, for $Y = \mathbb{R}$ and $\mathfrak{A}_Y = \mathfrak{M}_1$, f is a measurable function if $f^{-1}([a,b)) \in \mathfrak{A}_X$ for all $a, b \in \mathbb{R}$.

This will be shortened to saying that f is μ_X-measurable or even that f is measurable, if the underlying measure space is clear. In probability theory, such an f is called a **random variable**.

We will only be concerned with real-valued functions.

Theorem 44 *Let $(X, \mathfrak{A}, \mu)$ be a measure space and let $f : X \longrightarrow \mathbb{R}$ be a measurable function. The following are equivalent*

1. $f^{-1}([a, b)) \in \mathfrak{A}$
2. $f^{-1}((a, \infty)) \in \mathfrak{A}$
3. $f^{-1}((a, b)) \in \mathfrak{A}$
4. $f^{-1}([a, b]) \in \mathfrak{A}$
5. $f^{-1}((-\infty, a)) \in \mathfrak{A}$
6. $f^{-1}((-\infty, a]) \in \mathfrak{A}$

Proof. $(1 \Longrightarrow 2)$

To see this, note that

$$(a, \infty) = \bigcup_{n=1}^{\infty} (a, n) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left[a - \frac{1}{m}, n\right) \implies f^{-1}((a, \infty)) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} f^{-1}\left(\left[a - \frac{1}{m}, n\right)\right)$$

The set on the right hand-side is a countable intersection of a countable union of measurable sets, hence measurable. Therefore, $f^{-1}((a, \infty))$ is measurable. That is, $f^{-1}((a, \infty)) \in \mathfrak{A}$. The rest of the statements are verified in a similar manner. We only mention their decomposition.

$(2 \Longrightarrow 3)$

$(a, b) = (a, \infty) \cap (b, \infty)$

$(3 \Longrightarrow 4)$

$[a, b] = \bigcap_{n=1}^{\infty} \left(a + \frac{1}{n}, b + \frac{1}{n}\right)$

$(4 \Longrightarrow 5)$

$(-\infty, a) = \bigcup_{n=1}^{\infty} (-n, a) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} [1/m - n, a - 1/m]$

$(5 \Longrightarrow 6)$

$(-\infty, a] = \bigcap_{n=1}^{\infty} \left(-\infty, a + \frac{1}{n}\right)$

$(6 \Longrightarrow 1)$

We can use $(6 \Longrightarrow 5)$ because

$$(-\infty, a) = \bigcup_{n=1}^{\infty} \left(-\infty, a - \frac{1}{n}\right]$$

Taking complement on both sides gives us

$$[a,\infty)=\bigcap_{n=1}^{\infty}\left(a-\frac{1}{n},\infty\right).$$

Thus, assuming 6, we can have $(-\infty,b)$ for some $b>a$ and $(-\infty,b)\cap[a,\infty)=[a,b)\in\mathfrak{A}$. ■

Example 45 *Let $(X,\mathfrak{A},\mu)$ be a measure space and let $f:X\longrightarrow\mathbb{R}$ be defined by*

$$f(x)=\chi_E(x)=\begin{cases}1 & \text{if } x\in E\\ 0 & \text{if } x\notin E\end{cases}$$

In this case, f is measurable function if and only if $E\in\mathfrak{A}$: if f is measurable, then $f^{-1}(0,\infty)=E\in\mathfrak{A}$. Conversely, if E is measurable, then for any (a,b), if both 0 and 1 belong to (a,b), then $f^{-1}(a,b)=X$. If either belongs to (a,b), then $f^{-1}(a,b)=E^c$ or E, respectively. In either case, $f^{-1}(a,b)\in\mathfrak{A}$.

Problem 46 *Let $(X,\mathfrak{A},\mu)$ be a measure space, $E\in\mathfrak{A}$ and $\mu(E)>0$. Define $\mathfrak{A}_E:=\{A\subset E : A\in\mathfrak{A}\}$. Recall from **Problem 11** that $\mathfrak{A}_E$ is a σ-algebra and that $\mu|_E$ is a measure on $\mathfrak{A}_E$.*

1. *Prove that if f is measurable on X, then the restriction of f to E is measurable on E*

2. *Assume $\mu(X\backslash E)=0$ and $f:X\longrightarrow\mathbb{R}$ is a function such that its restriction to E is measurable with respect to $\mathfrak{A}_E$. Is it true that f is measurable on X? If no, what extra condition do we need for this to be true?*

Solution 47 ***1.*** *Let $f|_E=g$ and $I\subset\mathbb{R}$ be an interval. Then, $g^{-1}(I)=f^{-1}(I)\cap E$. Since $f^{-1}(I),E\in\mathfrak{A}$ (so that $f^{-1}(I)\cap E\in\mathfrak{A}$) and $f^{-1}(I)\cap E\subset E$, hence $f^{-1}(I)\cap E\in\mathfrak{A}_E$. That is, $g^{-1}(I)\in\mathfrak{A}_E$. Thus, g is measurable.*

2. *No. It may happen that $f^{-1}(A)\subset X\backslash E$ but $f^{-1}(A)\notin\mathfrak{A}$. For example, consider $X=\{1,2,3\}$ with $\mathfrak{A}=\{\varnothing,\{1,2\},\{3\},\{1,2,3\}\}$ and $E=\{3\}$. Define $\mu(\{1,2\})=0$ and $\mu(\{3\})=1$. Now define $f(x)=x$. Then, f restricted to E is a measurable function. However, $f^{-1}(1/2,3/2)=\{1\}\notin\mathfrak{A}$. For the extension of a measurable function to be measurable, we must have $\mathfrak{A}$ complete. That is, when every subset of a null set (a set of measure zero) is measurable. If $\mathfrak{m}_1(X\backslash E)=0$, then every subset of $X\backslash E$ is of measure zero and not included in $\mathfrak{A}_E$. Now, extend f by letting $f(x)\in f(E)^c$ if $x\in X\backslash E$. Then, either $f^{-1}(a,b)\subset E$ or $f^{-1}(a,b)\subset X\backslash E$. In the former case, $f^{-1}(a,b)\in\mathfrak{A}_E\subset\mathfrak{A}$ so that f is measurable. In the latter, $\mathfrak{m}_1\left(f^{-1}(a,b)\right)=0$ so that $f^{-1}(a,b)\in\mathfrak{A}$, making f again measurable.*

When is the composition of the two functions measurable? Continuous functions are always measurable so that gives us a lot but the following lemma shows we can generalise this a bit.

Lemma 48 *Let X be a topological space, $F : X \longrightarrow \mathbb{R}^p$ and $g : F(X) \longrightarrow \mathbb{R}$ be functions. If g is continuous and components of F are measurable, then $f = g \circ F$ is measurable.*

Proof. Let

$$f^{-1}((a,\infty)) = \{x : f(x) \in (a,\infty)\} = \left\{x : F(x) \in g^{-1}(a,\infty)\right\}$$

Then, $g^{-1}(a,\infty)$ is relatively open in $F(X)$. Thus, there exists an open set $O \subset \mathbb{R}^p$ such that $O \cap F(X) = g^{-1}(a,\infty)$. Since O is open, we can cover O by disjoint, open (even half-open work) rectangles $\{R_i : i \in \mathbb{N}\}$ in $\mathbb{R}^p$. Thus,

$$\begin{aligned}
f^{-1}(a,\infty) &= \{F(x) \in F(X) \cap O\} \\
&= \left\{F(x) \in F(X) \cap \bigcup_{i=1}^{\infty} R_i\right\} \\
&= \left\{x \in F^{-1}\left(\bigcup_{i=1}^{\infty} R_i\right)\right\} \\
&= \bigcup_{i=1}^{\infty} \left\{x \in F^{-1}(R_i)\right\}
\end{aligned}$$

Let us look at the individual pre-image of a rectangle, i.e. at $\left\{x \in F^{-1}(R_i)\right\}$. Since $R_i = I_1^i \times ... \times I_p^i$ and $F = \left(\varphi_1, ..., \varphi_p\right)$, with each component measurable by hypothesis, then $x \in F^{-1}(R_i) \iff x \in \varphi_k^{-1}(I_k)$ for each k so that

$$x \in \bigcap_{k=1}^{p} \varphi_k^{-1}(I_k) \implies f^{-1}((a,\infty)) = \bigcap_{k=1}^{p} \varphi_k^{-1}(I_k)$$

Thus, $f^{-1}((a,\infty)) \in \mathfrak{A}$. ■

Corollary 49 *If f, g are measurable functions, then $f+g, \alpha f, f.g, |f|, \max(f,g)$ and $\min(f,g)$ are measurable.*

Proof. Let f and g be two measurable functions defined on the same domain, X be a topological space, $F : X \longrightarrow \mathbb{R}^2$ defined by $F(x) = (f(x), g(x))$, $A : F(X) \times F(X) \longrightarrow \mathbb{R}$ defined by $A(x,y) = x + y$, $P : F(X) \times F(X) \longrightarrow \mathbb{R}$ defined by $P(x,y) = xy$, $G : X \longrightarrow \mathbb{R}$ defined by $G(x) = f(x)$, $S : F(X) \longrightarrow \mathbb{R}$ defined by $S(x) = \alpha x$, $M : F(X) \longrightarrow \mathbb{R}$ defined by $M(x) = |x|$, $M_{\mathrm{x}} : F(X) \times F(X) \longrightarrow \mathbb{R}^2$ defined by $M_{\mathrm{x}}(x,y) = \max(f(x), g(y))$ and $M_{\mathrm{n}} : F(X) \times F(X) \longrightarrow \mathbb{R}^2$ defined by $M_{\mathrm{n}}(x,y) = \min(f(x), g(y))$. Then, F and G have measurable functions as components and A, P, S, M, M_{x} and M_{n} are continuous functions. By above lemma, $A \circ F = f + g$, $P \circ F = f.g$, $S \circ G = \alpha f$, $M \circ G = |f|$, $M_{\mathrm{n}} \circ F = \min(f,g)$ and $M_{\mathrm{x}} \circ F = \max(f,g)$ are all measurable. ■

Theorem 50 *Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of measurable functions. Then, the following are measurable:*

$$\inf_{n\in\mathbb{N}} f_n(x)\,;\ \sup_{n\in\mathbb{N}} f_n(x)\,;\ \lim_{N\to\infty}\inf_{n\geq N} f_n(x)\ \text{ and } \lim_{N\to\infty}\sup_{n\geq N} f_n(x)$$

are all measurable.

Proof. First, let us call

$$f(x) = \sup_{n\in\mathbb{N}} f_n(x)$$

Consider,

$$f^{-1}((-\infty,a)) = \{x : f(x) < a\} = \left\{x : \sup_n f_n(x) < a\right\}$$

Thus, the set becomes

$$\{x : f_i(x) < a, \forall i \in \mathbb{N}\} = \bigcap_{i=1}^{\infty} \{x : f_i(x) < a\}$$

Now, $\{x : f_i(x) < a\} = f_i^{-1}((-\infty,a))$ is measurable, so we must have

$$\bigcap_{i=1}^{\infty} \{x : f_i(x) < a\} \in \mathfrak{A}$$

Thus, $f^{-1}((-\infty,a))$ is a measurable set and so f is a measurable function. Similarly, let

$$g(x) = \inf_{n\in\mathbb{N}} f_n(x)$$

Consider the set

$$\begin{aligned} g^{-1}((a,\infty)) &= \{x : g(x) > a\} = \left\{x : \inf_{n\in\mathbb{N}} f_n(x) > a\right\} \\ &= \{x : f_i(x) > a, \forall i \in \mathbb{N}\} = \bigcap_{i=1}^{\infty} \{x : f_i(x) > a\} \end{aligned}$$

Since $\{x : f_i(x)\} = f_i^{-1}((a,\infty))$ is measurable, we must have

$$\bigcap_{i=1}^{\infty} \{x : f_i(x) > a\} \in \mathfrak{A}$$

Thus, $g^{-1}((a,\infty))$ is a measurable set so that

$$g = \inf_{n\in\mathbb{N}} f_n$$

is a measurable function. Since

$$\inf_{n\geq N} f_n(x)$$

is an increasing sequence, we can have

$$\lim_{N\to\infty}\inf_{n\geq N} f_n(x) = \sup_N \inf_{n\geq N} f_n(x)$$

Since $f_i(x)$ is measurable for each i, then

$$g_N(x) = \inf_{n\geq N} f_n(x)$$

is measurable so that

$$\sup_{N\in\mathbb{N}} g_N(x)$$

is measurable. Now, recall that

$$\lim_{N\to\infty}\sup_{n\geq N} f_n(x) = \inf_{N\geq 0}\sup_{n\geq N} f_n(x)$$

Then, $f_n(x)$ is measurable
$\Longrightarrow \sup_{n\geq N} f_n(x)$ is measurable
$\Longrightarrow \inf_{N\geq 0}\sup_{n\geq N} f_n(x)$ is measurable ■

Corollary 51 *If $\{f_n : n \in \mathbb{N}\}$ is a sequence of measurable functions, and $f_n \longrightarrow f$ pointwise (i.e. $f_n(x) \longrightarrow f(x)$ for every $x \in X$), then $f(x)$ is measurable.*

Proof. Since $f_n(x) \longrightarrow f(x)$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{N\to\infty}\sup_{n\geq N} f_n(x) = \lim_{N\to\infty}\inf_{n\geq N} f_n(x)$$

The latter two are measurable. ■

What if $f_n(x) \longrightarrow f(x)$ almost everywhere? Under certain conditions, $f(x)$ is measurable:

Problem 52 *Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of real-valued measurable functions on X. For every natural n, define*

$$E_n := \left\{x \in X : |f_n(x) - f_{n+1}(x)| > 2^{-n}\right\}$$

Show that if $\mu(E_n) < 2^{-n}$ for every n, then f_n is pointwise convergent almost everywhere on X.

Solution 53 *Since $\mu(E_n) < 2^{-n}$ for every n, we have a family of measurable sets $\{E_n : n \in \mathbb{N}\}$ and their union has finite measure,*

$$\sum_{n=1}^{\infty}\mu(E_n) < \sum_{n=1}^{\infty}\frac{1}{2^n} = 1 < \infty$$

Then, by Borel-Cantelli Lemma, almost every $x \in X$ belongs to at most finitely many E_n's. In other words, almost no $x \in X$ belongs to infinitely many E_n's. By definition of E_n, f_n does not converge pointwise on E_n since for $\epsilon = 2^{-n}$, there exists no N such that $|f_n(x) - f_{n+1}(x)| < 2^{-n}$ for $N \geq n$. Thus, for almost every $x \in X$, f_n is not pointwise convergent on at most finitely many E_n's, which means that f_n is pointwise convergent almost everywhere.

2.2 Simple Functions

Let us now approximate! For measurable functions, there are many canonical ways that work but the one with a simple function is what we will study.

Definition 54 *A function $f : X \longrightarrow \mathbb{R}$ is called **simple** if there exist measurable sets $E_1, ..., E_n$ such that*

$$f(x) = \sum_{j=1}^{n} c_j . \chi_{E_J}(x)$$

where c_j are scalars.

These are also called **step functions**. The measurable sets are not required to be disjoint. These are called simple because they're easy to integrate.

Problem 55 *Show that a sum, a product, a min and a max of two simple functions is a simple function.*

Solution 56 *Let $f : X \longrightarrow \mathbb{R}$ and $g : X \longrightarrow \mathbb{R}$ be simple. Then, $\exists$ measurable sets $E_1, ..., E_n$ and $F_1, ..., F_m$ such that*

$$f = \sum_{j=1}^{n} c_j . \chi_{E_j} \text{ and } g = \sum_{i=1}^{m} d_i . \chi_{F_i}$$

where $E_j = \{x \in X : f(x) = c_j\}$ and $F_i = \{x \in X : g(x) = d_i\}$. We can assume that $c_i \neq c_j$ for $i \neq j$ so that $E_i \neq E_j$. Similarly for g.

Since the intersection of measurable sets is measurable, we have $E_j \cap F_i$ is measurable for $1 \leq j \leq n$ and $1 \leq k \leq m$.

Recall that the domain for the sum and product of two functions is the intersection of the domains. Also recall that $\chi_{E_i \cap F_j} = \chi_{E_i} + \chi_{F_j} - \chi_{E_i}\chi_{F_j}$. Then, if we let $c_k + d_k = a_k$ and $c_k = 0$ and $E_k = \varnothing$ for $k > n = \min\{m, n\}$ or $d_k = 0$ and $F_k = \varnothing$ for $k > m = \min\{m, n\}$, then

$$f+g = \sum_{i=1}^{\max\{m,n\}} a_i \chi_{E_i \cap F_i} = \sum_{i=1}^{\max\{m,n\}} a_i \chi_{E_i} + \sum_{i=1}^{\max\{m,n\}} a_i \chi_{F_i} - \sum_{i=1}^{\max\{m,n\}} a_i \chi_{E_i} \chi_{F_i} < \infty$$

Moreover

$$\max(f, g) = \sum_{i=1}^{\max\{m,n\}} \max(c_i, d_i) \chi_{E_i \cap F_j}$$

and,

$$\min(f, g) = \sum_{i=1}^{\max\{m,n\}} \min(c_i, d_i) \chi_{E_i \cap F_j}$$

are both finite so that the sum, product, min and maximum of simple functions is simple. Similarly, for $f.g$ where $(f.g)(x) = f(x) g(x)$, if we let $c_k d_k = b_k$

and $c_k = 1$ *and* $E_k = \varnothing$ *for* $k > n = \min\{m,n\}$ *or* $d_k = 1$ *and* $F_k = \varnothing$ *for* $k > m = \min\{m,n\}$*, then*

$$f.g = \sum_{i=1}^{\max\{m,n\}} b_i \chi_{E_i \cap F_j} < \infty$$

2.2.1 Approximation of Measurable Functions

Lemma 57 *Assume that* $f : X \longrightarrow \mathbb{R}$ *is measurable and bounded (i.e.* $|f(x)| \leq M$ *for every* $x \in X$*). Then,* $\forall \epsilon > 0$*,* $\exists$ *simple* φ, ψ *with* $\varphi \leq f \leq \psi$ *such that* $|\varphi(x) - \psi(x)| < \epsilon\ \forall x$

Proof. Let $\epsilon > 0$ and $\forall x$, we know that $f(x) \in [-M, M+1)$. Denote $c = -M$ and $d = M+1$. Take $y_0 = c < y_1 < ... < y_n = d$ such that $y_k - y_{k-1} < \epsilon$ for each k. Then, each $X_k = f^{-1}[y_{k-1}, y_k)$ is measurable. Note that X_k's are disjoint. Define

$$\varphi(x) = \sum_{k=1}^{n} y_{k-1} . \chi_{X_k}(x)$$

and

$$\psi(x) = \sum_{k=1}^{n} y_k . \chi_{X_k}(x)$$

Since $f(x) \in [c,d)$, $\exists! k$ such that $y_{k-1} \leq f(x) \leq y_k$. That is, if $x \in X_k$, then $\varphi(x) = y_{k-1} \leq f(x) \leq y_k = \psi(x)$. Thus, $\varphi \leq f \leq \psi$. Moreover, since $|y_{k-1} - y_k| < \epsilon$, we must have $|\varphi(x) - \psi(x)| < \epsilon\ \forall x$. ■

Theorem 58 *Let* $f : X \longrightarrow \mathbb{R}$ *be measurable. Then, there is a sequence* $\{\psi_n\}$ *of simple functions such that*

1. $|\psi_n(x)| \leq |f(x)|$
2. $\psi_n(x) \to f(x)$ *for every* $x \in X$
3. *If* $f \geq 0$*, then* $\psi_n(x) \nearrow f(x)$*. That is, for a fixed* x*, the sequence* ψ_n *is increasing and converges to* $f(x)$*.*

Proof. Let $E_n = \{x \in X : |f(x)| \leq n\}$. Then, E_n is measurable. By **Lemma 57**, $\exists g_n, h_n$ in E_n with $g_n \geq f \geq h_n$ and $|g_n - h_n| < \frac{1}{n}$. For $x \in E_n$, define

$$\psi_n(x) = \begin{cases} 0 & f(x) = 0 \\ \max(h_n(x), 0) & f(x) > 0 \\ \min(g_n(x), 0) & f(x) < 0 \end{cases}$$

and for $x \notin E_n$, set $\psi_n(x) = 0$. This is a simple function since it has finitely many values and is absorbed in the characteristic function. The values of $\psi_n(x)$ are obtained by measurable functions; thus ψ_n is measurable, for each n.

Now we prove (1). Fix an $x \in X$. If $x \in E_n$ and $f(x) > 0$, then $h_n(x) > 0$ so that $h_n \leq f \Longrightarrow |\psi_n(x)| \leq |f(x)|$. If $x \in E_n$ and $f(x) < 0$, then $g_n(x) < 0$ so that $f \leq g_n < 0 \Longrightarrow |\psi_n(x)| \leq |f(x)|$. In all other cases, $\psi_n(x) = 0$ so $|\psi_n(x)| \leq |f(x)|$.

For (2), there are three cases to consider.

If $0 < f(x) < \infty$ for every x, then $\exists N$ such that $|f(x)| \leq N$ so that for $n \geq N$ and $x \in E_n$, $0 \leq \psi_n(x) = \max(h_n(x), 0) \leq |f(x)|$ so that from $0 \leq h_n(x) = \psi_n(x) \leq f(x) \leq g_n(x)$ and $0 \leq g_n(x) - h_n(x) < \frac{1}{n}$, we get $0 \leq f(x) - \psi_n(x) \leq g(x) - h_n(x) < \frac{1}{n}$. Letting $n \to \infty$, we get $f(x) - \psi_n(x) \to 0$ so that $\psi_n(x) \to f(x)$.

If $0 > f(x) > -\infty$ for every x, then $\exists N$ such that $|f(x)| \leq N$ so that for $n \geq N$ and $x \in E_n$, $0 \leq \psi_n(x) = \min(g_n(x), 0) \leq |f(x)|$ so that $0 \leq f(x) - \psi_n(x) \leq f(x) - g_n(x) < \frac{1}{n}$. Thus, $f(x) - \psi_n(x) \to 0$.

If $f(x) = \pm\infty$, then define

$$\widetilde{\psi}_n(x) = \begin{cases} \psi_n(x) & x \in E_n \\ n & f(x) > n \\ -n & f(x) < -n \end{cases}$$

Then, by definition,

$$\left|\widetilde{\psi}_n(x)\right| \leq |f(x)|$$

and $\widetilde{\psi}_n \to f$.

For (3), we need special consideration since the sequence ψ_n is not necessarily increasing. This is because the approximations h_n and g_n, as constructed in **Lemma 57**, rely on the nature of f itself. Thus, h_n's and g_n's are not necessarily increasing since f is not. We get our way around: if $f \geq 0$, then define

$$\overline{\psi}_n(x) = \max_{1 \leq j \leq n} \widetilde{\psi}_j(x)$$

Clearly, $\overline{\psi}_n$ is an increasing sequence of functions. Moreover, since

$$\left|\widetilde{\psi}_n(x)\right| \leq |f(x)|$$

for each n, we must have

$$\left|\overline{\psi}_n(x)\right| \leq |f(x)|$$

Finally, it is easy to see that $0 \leq f(x) - \overline{\psi}_n(x) \leq f(x) - \widetilde{\psi}_n(x) \to 0$.

Since h_n and g_n are simple, then so is ψ_n, and so is $\widetilde{\psi}_j$, and, therefore, $\overline{\psi}_n$ is also a simple function for each n. ∎

Lemma 59 *Let $f : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a (measurable) function. Let $E \in \mathfrak{M}_p$, $\mathfrak{m}_p(E) < \infty$. Assume that $\{f_n : n \in \mathbb{N}\}$ is a sequence of measurable functions with $f_n(x) \to f(x)$ $\forall x \in E$ (i.e. pointwise). Then, $\forall \epsilon > 0$, $\delta > 0$, there exists $A \subset E$ with $A \in \mathfrak{M}_p$ such that $|f_n - f| < \epsilon$ on A $\forall n \geq N$ and $\mathfrak{m}_p(E \backslash A) < \delta$*

A is not necessarily closed. Hence it differs from what we have established in **Theorem 35**. That is, the existence of a closed set A such that $\mathfrak{m}_p(E\backslash A) < \delta$. Although this doesn't necessarily matter, as will be clear in the proof.
Proof. Note that f_n is measurable and

$$f = \lim_{n\to\infty} f_n = \lim_{N\to\infty} \sup_{n\geq N} f_n = \lim_{n\to\infty} \inf_{n\geq N} f_n$$

so that f is also measurable, hence the brackets in the hypothesis. Define

$$E_n = \{x \in \mathbb{R}^p : |f(x) - f_k(x)| < \epsilon \ \forall k \geq n\}$$

This set is measurable because both f_n and f are measurable. That is, $E_n \in \mathfrak{M}_p$. Note that $E_n \subset E_{n+1}$ for each $n \geq 1$. Define

$$\bigcup_{j=1}^{\infty} E_j = E$$

so that

$$\mathfrak{m}_p(E) = \lim_{n\to\infty} \mathfrak{m}_p(E_n)$$

Hence $\exists N$ such that $\forall n > N$, $\mathfrak{m}_p(E) < \mathfrak{m}_p(E_n) + \delta$ for $n \geq N$ and, so, $\mathfrak{m}_p(E\backslash E_n) < \delta$ for $n \geq N$. Setting $A = E_{N+1}$ establishes the lemma. ■

Theorem 60 (Egoroff) *Let $f : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a measurable function. Let $E \in \mathfrak{M}_p$, $\mathfrak{m}_p(E) < \infty$. Assume that $\{f_n : n \in \mathbb{N}\}$ is a sequence of measurable functions such that $\forall x \in E$, $f_n(x) \to f(x)$ pointwise. Then, $\exists$ a closed $F \subset E$ with $\mathfrak{m}_p(E\backslash F) < \epsilon$ and $f_n \to f$ uniformly on F.*

This theorem holds in more generality for measure spaces. Thus, up to a measure ϵ, pointwise convergence implies uniform convergence.
Proof. By **Lemma 59**, for a fixed n, we can find $A_n \subset E$ and $N(n)$ such that $\mathfrak{m}_p(E\backslash A_n) < \frac{\varepsilon}{2^{n+1}}$ for $n \geq N$ so that $\forall k \geq N(n)$, $|f - f_k| < \frac{1}{n}$ on A_n. Now let

$$A = \bigcap_{j=1}^{\infty} A_j$$

A may not be closed! Then,

$$\begin{aligned}
\mathfrak{m}_p(E\backslash A) &= \mathfrak{m}_p\left(\bigcup_{n=1}^{\infty}(E\backslash A_n)\right) \\
&= \sum_{n=1}^{\infty}\mathfrak{m}_p(E\backslash A_n) \\
&< \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}
\end{aligned}$$

Our goal now is to show that $f_n \to f$ on A uniformly. This is written as $f_n \rightrightarrows f$ on A. That is, $\forall \sigma > 0$, $\exists M$ such that $\forall k > M$, $|f_k(x) - f(x)| < \sigma$ for every $x \in A$. Take n such that $\frac{1}{n} < \sigma$. Then, for every $x \in A_n$, we have $|f_k(x) - f(x)| < \frac{1}{n} < \sigma$ as soon as $k \geq N(n) = N(\sigma)$. Let $M = N(n) = N(\sigma)$. Then, for a fixed $\sigma > 0$, we found an M such that $\forall k \geq M$, $\forall x \in A$, we have $|f_k(x) - f(x)| < \sigma$, i.e. $f_n \rightrightarrows f$.

Now, recall that $\mathfrak{m}_p(E \backslash A) < \frac{\varepsilon}{2}$. By the Regularity of the Lebesgue Measure (i.e. $\mathfrak{m}_p(E \backslash A) = \sup\{\mathfrak{m}_p(K) : K \text{ is compact and } K \subset E \backslash A\}$) we can find a closed $F \subset A$ with $\mathfrak{m}_p(A \backslash F) < \frac{\varepsilon}{2}$. Then, $\mathfrak{m}_p(E \backslash F) < \varepsilon$. ■

Problem 61 *Show that the conclusion of Egoroff's Theorem can fail if we drop the assumption that the domain has finite measure.*

Solution 62 *Consider the sequence of indicator functions* $\chi_{[-n,n]} := f_n$ *defined on* $\mathbb{R}$*. Then, for* $x \in \mathbb{R}$*,*

$$\lim_{N \to \infty} \sup_{n \geq N} f_n = \lim_{n \to \infty} \inf_{n \geq N} f_n = 1$$

so that f_n *converges to the constant function* $\mathbf{1}$*. That is, to a function* $\mathbf{1} : \mathbb{R} \longrightarrow \mathbb{R}$ *such that* $\mathbf{1}(x) = 1$ *for all* x*, pointwise. However, assume that* $f_n \rightrightarrows \mathbf{1}$ *on some closed subset* F *of* $\mathbb{R}$*. Choose* $\epsilon = 1$*, then for* $k \in \mathbb{N}$*,* $|\mathbf{1}(x) - f_k(x)| < 1$ *holds when* $f_k(x) = 1$ *so that, for the* N *we should be able to find, given our* $\epsilon = 1$*, for* $n \geq N$*,*

$$E_n = \{x \in \mathbb{R} : |\mathbf{1} - f_k(x)| < 1 \ \forall k \geq n\} = \bigcap_{k=N}^{\infty} [-k, k] = [-N, N] = F$$

However, $\mathfrak{m}_1(\mathbb{R} \backslash F) = \mathfrak{m}_1((-\infty, N) \cup (N, \infty)) \nless 1$*.*

Lemma 63 *Let* $E \subset \mathbb{R}^p$ *and* $f : E \longrightarrow \mathbb{R}$ *be a simple function. Then,* $\forall \epsilon > 0$*,* $\exists$ *a continuous function* $g : \mathbb{R}^p \longrightarrow \mathbb{R}$ *and a closed set* $F \subset E$ *such that* $\mathfrak{m}_p(E \backslash F) < \epsilon$*,* $f(x) = g(x)$ *if* $x \in F$

Proof. Let

$$f(x) = \sum_{k=1}^{n} a_k \chi_{E_k}(x)$$

where we can assume WLOG that E_k are disjoint. Then, we can find a closed set $F_k \subset E_k$ such that $\mathfrak{m}_p(E_k \backslash F_k) < \frac{\epsilon}{2^k}$. Set

$$F = \bigcup_{k=1}^{n} F_k$$

Note the finite union: this ensures that F is closed. Moreover, $\mathfrak{m}_p(E \backslash F) < \epsilon$. Set $g(x) = a_k$ when $x \in F_k$. g is defined on F and it is continuous on F: if $x \in F$, then $\exists! k'$ such that $x \in F_{k'}$. That is, some neighborhood of x does not intersect F_j for $j \neq k'$. Now, the continuity of g is easy to prove. We can now extend g to a continuous function on $\mathbb{R}^p$ by setting constant values for the "end points" of F. ■

Problem 64 *Define*

$$f(x) := \begin{cases} 2 & \text{if } x \in [0,1) \\ 5 & \text{if } x \in [1,2) \\ -1 & \text{if } x \in [3,4) \\ 0 & \text{otherwise} \end{cases}$$

For every $\epsilon > 0$, construct a continuous function $\varphi_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mathfrak{m}_1(\{x : f(x) \neq \varphi_\epsilon(x)\}) < \epsilon$.

Solution 65 *Let $E_1 = [0,1)$, $E_2 = [1,2)$ and $E_3 = [3,4)$ and $E_4 = [4,\infty)$. Then, $f = 2\chi_{E_1} + 5\chi_{E_2} - \chi_{E_3} + 0\chi_{E_4}$ is simple and, therefore, measurable. Thus, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a simple function, so we are guaranteed the existence of such a continuous function by **Lemma 63**. We can choose the closed sets F_k to be*

$$F_1 = \left[\frac{\epsilon}{4}, 1 - \frac{\epsilon}{4}\right], F_2 = \left[1, 2 - \frac{\epsilon}{2^2}\right], F_3 = \left[3 + \frac{\epsilon}{2^4}, 4 - \frac{\epsilon}{2^4}\right] \text{ and } F_4 = \left[4 + \frac{\epsilon}{2^4}, \infty\right)$$

Then, for $F = F_1 \cup F_2 \cup F_3 \cup F_4$ and $E = E_1 \cup E_2 \cup E_3 \cup E_4$,

$$\mathfrak{m}_1(E \backslash F) = \frac{\epsilon}{2} + \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \frac{\epsilon}{2^4} = \frac{15}{16}\epsilon < \epsilon$$

Now, define $\varphi_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$

$$\varphi_\epsilon(x) = \begin{cases} \left(\frac{8}{\epsilon} - 2\right)x & \text{if } x \in \left(0, \frac{\epsilon}{4}\right) \\ 2 & \text{if } x \in F_1 \\ \frac{12}{\epsilon}x + 2 & \text{if } x \in \left(1 - \frac{\epsilon}{4}, 1\right) \\ 5 & \text{if } x \in F_2 \\ \frac{40}{\epsilon} - \frac{20}{\epsilon}x & \text{if } x \in \left(2 - \frac{\epsilon}{4}, 2\right) \\ \frac{48}{\epsilon} - \frac{16}{\epsilon}x & \text{if } x \in \left(3, 3 + \frac{\epsilon}{16}\right) \\ -1 & \text{if } x \in F_3 \\ \frac{8}{\epsilon}x - \frac{1}{2} - \frac{32}{\epsilon} & \text{if } x \in \left(4 - \frac{\epsilon}{16}, 4 + \frac{\epsilon}{16}\right) \\ 0 & \text{otherwise} \end{cases}$$

By construction, φ_ϵ is continuous and $E \backslash F = \{x : f(x) \neq \varphi_\epsilon(x)\}$

Theorem 66 (Lusin) *Let $E \subset \mathbb{R}^p$ and $f : E \longrightarrow \mathbb{R}$ be a measurable function. Then, $\forall \epsilon > 0$, $\exists$ a continuous function $g : \mathbb{R}^p \longrightarrow \mathbb{R}$ and a closed set $F \subset E$ such that $\mathfrak{m}_p(E \backslash F) < \epsilon$, $f(x) = g(x)$ $\forall x \in F$*

That is, a measurable function can be approximated by a continuous function.

Proof. There are two cases to consider, with E having finite and infinite measure.

Assume that $\mathfrak{m}_p(E) < \infty$.

Since f is measurable, $\exists$ a sequence of simple functions $\{\varphi_n : n \in \mathbb{N}\}$ on E such that $\varphi_n \to f$. As shown in **Problem 64** and proved in **Lemma 63**, for each n, we can find a continuous function g_n such that $F_n \subset E$ is closed and

$$\mathfrak{m}_p(E \backslash F_n) < \frac{\epsilon}{2^{n+1}}$$

Note that $g_n = \varphi_n$ on F_n. By Egoroff, there is a closed $F_0 \subset E$ where $\varphi_n \rightrightarrows f$ and

$$\mathfrak{m}_p \left(E \backslash F_0\right) < \frac{\epsilon}{2}$$

Now, let

$$F = \bigcap_{n=0}^{\infty} F_n$$

where F_n is closed and $\mathfrak{m}_p \left(E \backslash F\right) < \epsilon$. On F, we have $g_n \left(x\right) \overset{x \in F_n}{=} \varphi_n \left(x\right) \rightrightarrows f \left(x\right)$. That is, on F, $\{g_n : n \in \mathbb{N}\}$ converges uniformly to f. Then, f is continuous on F. We can then set $g = f|_F$. Then, g is continuous on F. We can now extend g to $\mathbb{R}^p$.

If $\mathfrak{m}_p \left(E\right) = \infty$, we can split E into boxes of side-length 1. Let

$$E = \bigcup_{i=0}^{\infty} B_i$$

and let $E_i = E \cap B_i$. Then,

$$E = \bigcup_{i=0}^{\infty} E_i$$

and E_i for each i is measurable, $\mathfrak{m}_1 \left(E_i\right) \leq 1$ and by **Problem 46**, f is measurable on E_i and, therefore, $\exists$ a sequence of simple functions $\{\varphi_n : n \in \mathbb{N}\}$ on E_i such that $\varphi_n \to f$. Again, by **Lemma 63**, for each n, we can find a continuous function g_n such that $F_n^{(i)} \subset E_i$ is closed and

$$\mathfrak{m}_p \left(E_i \backslash F_n^{(i)}\right) < \frac{\epsilon}{2^{i+n+1}}$$

with $g_n = \varphi_n$ on $F_n^{(i)}$. By Egoroff, there is a closed $F_0^{(i)} \subset E_i$ where $\varphi_n \rightrightarrows f$ and

$$\mathfrak{m}_p \left(E_i \backslash F_0^{(i)}\right) < \frac{\epsilon}{2^{i+1}}$$

Now, let

$$F^{(i)} = \bigcap_{n=0}^{\infty} F_n^{(i)}$$

where $F_n^{(i)}$ is closed and $\mathfrak{m}_p \left(E_i \backslash F^{(i)}\right) < \frac{\epsilon}{2^{i+1}}$. On $F^{(i)}$, we have $g_n \left(x\right) \overset{x \in F_n^{(i)}}{=} \varphi_n \left(x\right) \rightrightarrows f \left(x\right)$. That is, on $F^{(i)}$, $\{g_n : n \in \mathbb{N}\}$ converges uniformly to f. Then, f is continuous on $F^{(i)}$. We can then set $g_i = f|_{F^{(i)}}$. Then, g_i is continuous on $F^{(i)}$. Now let

$$h \left(x\right) = \sum_{i=1}^{\infty} g_i \left(x\right) \chi_{F^{(i)}} \left(x\right) \text{ and } F' = \bigcup_{i=0}^{\infty} F^{(i)} \subset \bigcup_{i=0}^{\infty} E_i$$

Then, $g = f|_F$. This union, however, may not be closed but is, however, measurable. Thus, we can find a closed set $F \subset F'$ and by Egoroff, a continuous function g such that $\mathfrak{m}_1 \left(F' \backslash F\right) < \epsilon/2$ so that $\mathfrak{m}_1 \left(E \backslash F'\right) < \epsilon$ and $g \left(x\right) = h \left(x\right)$ on F. ■

3 Integration

3.1 Integral of Simple and Measurable Functions

The integral is naturally defined using simple functions over a set of finite measure. To do this, however, we need to ensure that the integral does not depend on the representation of the simple function:

Lemma 67 *Let* $(X, \mathfrak{A}, \mu)$ *be a measure space and let* $\varphi : X \longrightarrow \mathbb{R}$ *be a simple function. Assume that*

$$\varphi(x) = \sum_{j=1}^{N} a_j \chi_{A_j}(x) = \sum_{k=1}^{M} b_k \chi_{B_k}(x)$$

where $\{A_j : 1 \le j \le N\}$ *and* $\{B_j : 1 \le j \le M\}$ *are both pair-wise disjoint and measurable, and that*

$$\bigcup_{j=0}^{N} A_j = X = \bigcup_{j=0}^{M} B_j$$

Then, for every $E \in \mathfrak{A}$*,*

$$\sum_{j=1}^{N} a_j \mu(A_j \cap E) = \sum_{k=1}^{M} b_k \mu(B_k \cap E)$$

Proof. Note that $a_j \mu(A_j \cap E \cap B_k) = b_k \mu(B_k \cap E \cap A_j)$ for each j, k. This is because if A_j and B_k are disjoint, then we trivially have equality on both sides since we then have $0 = 0$. If $x \in A_j \cap B_k$, then let $\varphi(x) = c_j$. Since $x \in A_j$, we have $\varphi(x) = a_j$ and $\varphi(x) = b_k$ because $x \in B_k$. Thus, $c_j = b_k = a_j$. Now, $\forall k$,

$$B_k = B_k \cap X = B_k \cap \left(\bigcup_{j=1}^{N} A_j \right) = \bigcup_{j=1}^{N} (A_j \cap B_k)$$

and

$$\begin{aligned}
\sum_{k=1}^{M} b_k \mu(B_k \cap E) &= \sum_{k=1}^{M} b_k \mu\left(E \cap \bigcup_{j=1}^{N} (A_j \cap B_k)\right) = \sum_{k=1}^{M} b_k \mu\left(\bigcup_{j=1}^{N} (E \cap A_j \cap B_k)\right) \\
&= \sum_{k=1}^{M} b_k \sum_{j=1}^{N} \mu(E \cap A_j \cap B_k) = \sum_{k=1}^{M}\sum_{j=1}^{N} b_k \mu(E \cap A_j \cap B_k) \\
&= \sum_{k=1}^{M}\sum_{j=1}^{N} a_j \mu(E \cap A_j \cap B_k) = \sum_{j=1}^{N} a_j \sum_{k=1}^{M} \mu(E \cap A_j \cap B_k) \\
&= \sum_{j=1}^{N} a_j \mu\left(\bigcup_{k=1}^{M} (E \cap A_j \cap B_k)\right) = \sum_{j=1}^{N} a_j \mu\left(E \cap \bigcup_{k=1}^{M} (A_j \cap B_k)\right) \\
&= \sum_{j=1}^{N} a_j \mu\left(E \cap \left(A_j \cap \bigcup_{k=1}^{M} B_k\right)\right) = \sum_{j=1}^{N} a_j \mu(E \cap (A_j \cap X)) \\
&= \sum_{j=1}^{N} a_j \mu(E \cap A_j)
\end{aligned}$$

■

Now, assume that we have a simple, positive function φ, on X with

$$\varphi(x) = \sum_{i=1}^{N} a_i \chi_{A_i}(x)$$

where $\{A_i : 1 \leq i \leq N\}$ is pair-wise disjoint and measurable. Then for every $E \in \mathfrak{A}$, we define the integral of φ over E with respect to μ as follows:

$$\int_E \varphi d\mu = \sum_{i=1}^{N} a_j \mu(A_i \cap E)$$

Lemma 68 *Let φ be a simple, non-negative function with*

$$\varphi(x) = \sum_{i=1}^{N} a_i \chi_{A_i}(x)$$

Let μ be a measure and let $E \in \mathfrak{A}$. Then,

$$\int_E \varphi d\mu \in [0, \infty]$$

Proof. Since

$$\int_E \varphi d\mu = \sum_{i=1}^{N} a_j \mu(A_i \cap E)$$

and each $a_j > 0$ by hypothesis ($\mu \geq 0$ by default) ■

Lemma 69 *Let φ be a constant simple, non-negative function. That is, $\varphi(x) = c\ \forall x \in E$ where $E \in \mathfrak{A}$. Let μ be a measure. Then,*

$$\int\limits_E \varphi d\mu = c\mu(E)$$

Proof. If $\varphi(x) = c$, then $\varphi(x) = c\chi_E$ so that

$$\int_E \varphi d\mu = c\mu(E \cap E) = c\mu(E)$$

■

Lemma 70 *Let φ and ψ be simple, non-negative functions with*

$$\varphi(x) = \sum_{i=1}^{N} a_i \chi_{A_i}(x) \ \text{ and } \psi(x) = \sum_{i=1}^{M} b_i \chi_{B_i}(x)$$

Let μ be a measure, α, β be scalars and $E \in \mathfrak{A}$. Then,

$$\int_E (\alpha\varphi + \beta\psi) d\mu = \alpha \int_E \varphi d\mu + \beta \int_E \psi d\mu$$

Proof. The result will be proven in two steps: first, we will show that the sum gets distributed and later on, that the scalar product can be pulled out.

We may choose a finite, disjoint collection C_i such that

$$\varphi(x) = \sum_{i=1}^{K} \alpha_i \chi_{C_i}(x) \text{ and } \psi(x) = \sum_{i=1}^{K} \beta_i \chi_{C_i}(x)$$

Then,

$$(\varphi + \psi)(x) = \sum_{i=1}^{K} (\alpha_i + \beta_i) \chi_{C_i}(x)$$

so that

$$\begin{aligned}
\int_E (\varphi + \psi) d\mu &= \sum_{i=1}^{K} (\alpha_i + \beta_i) \mu(C_i \cap E) \\
&= \sum_{i=1}^{K} \alpha_i \mu(C_i \cap E) + \sum_{i=1}^{K} \beta_i \mu(C_i \cap E) \\
&= \int_E \varphi d\mu + \int_E \psi d\mu
\end{aligned}$$

Now,

$$\int_E \alpha\varphi d\mu = \sum_{i=1}^{K} \alpha(\alpha_i \mu(C_i \cap E)) = \alpha \sum_{i=1}^{K} (\alpha_i \mu(C_i \cap E)) = \alpha \int_E \varphi d\mu$$

Combining these two facts readily allows for the required result. ■

Lemma 71 *Let φ and ψ be simple, non-negative functions with*

$$\varphi(x)=\sum_{i=1}^{N} a_i\chi_{A_i}(x) \text{ and } \psi(x)=\sum_{i=1}^{M} b_i\chi_{B_i}(x)$$

Let μ be a measure, $\varphi \leq \psi$ and let $E \in \mathfrak{A}$. Then,

$$\int_E \varphi d\mu \leq \int_E \psi d\mu$$

Proof. Again, we may choose a finite, disjoint collection C_i such that

$$\varphi(x)=\sum_{i=1}^{K} \alpha_i\chi_{C_i} \text{ and } \psi(x)=\sum_{i=1}^{K} \beta_i\chi_{C_i}$$

with $\alpha_i \leq \beta_i$ so that

$$\int_E \varphi d\mu=\sum_{i=1}^{K} \alpha_i\mu(C_i\cap E) \leq \sum_{i=1}^{K} \beta_i\mu(C_i\cap E)=\int_E \psi d\mu$$

■

Lemma 72 *Let φ be a simple, non-negative function with*

$$\varphi(x)=\sum_{i=1}^{N} a_i\chi_{A_i}(x)$$

Let μ be a measure and let $E \in \mathfrak{A}$. Then,

$$\int_E \varphi d\mu=\int_X \varphi\chi_E d\mu$$

Proof. Note that $\varphi\chi_E$ is simple and is equal to a_j on $E\cap A_j$ and 0 on E^c. Also, note that $E\cap A_1, E\cap A_2, ..., E\cap A_k, E^c$ form a partition of X. Then,

$$\int_X \varphi\chi_E d\mu=\sum_{j=1}^{N} a_j\mu(A_j\cap E)+0\mu(E^c)=\int_E \varphi d\mu+0=\int_E \varphi d\mu$$

■

Lemma 73 *Let φ be a simple, non-negative function with*

$$\varphi(x)=\sum_{i=1}^{N} a_i\chi_{A_i}$$

Let μ be a measure and let $E, F \in \mathfrak{A}$ with $E \subset F$. Then,

$$\int_E \varphi d\mu \leq \int_F \varphi d\mu$$

Proof. Follows from **Lemmas 71** and **72** because $\varphi\chi_E \leq \psi\chi_F$ on X. ■

Lemma 74 *Let φ be a simple, non-negative function with*

$$\varphi(x) = \sum_{i=1}^{N} a_i \chi_{A_i}(x)$$

Let μ be a measure and let $E \in \mathfrak{A}$. Then

$$\int_E \varphi d\mu = \sup\left\{\int_E \psi d\mu : \psi \text{ simple, } \psi \geq 0, \psi \leq \varphi \text{ on } E\right\}$$

Proof. By **Lemma 71**

$$\int_E \psi d\mu \leq \int_E \varphi d\mu \ \forall 0 \leq \psi \leq \varphi$$

hence

$$\sup\left\{\int_E \psi d\mu : \psi \text{ simple, } \psi \geq 0, \psi \leq \varphi \text{ on } E\right\} \leq \int_E \varphi d\mu$$

But $\varphi = \psi$ is a candidate, as well. Hence

$$\sup\left\{\int_E \psi d\mu : \psi \text{ simple, } \psi \geq 0, \psi \leq \varphi \text{ on } E\right\} \geq \int_E \varphi d\mu$$

establishing equality. ■

Lemma 74 hints at what the definition of the integral of a measurable function should be, since we can approximate a measurable function by a sequence of simple functions.

Definition 75 *Let f be a measurable and non-negative function. Then, the* ***integral*** *of f is defined as*

$$\int_E f d\mu = \sup\left\{\int_E \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f \text{ on } E\right\}$$

Proposition 76 *Let $(X, \mathfrak{A}, \mu)$ be a measure space, $E \in \mathfrak{A}$ and let $f : E \longrightarrow \mathbb{R}$ be a measurable and non-negative function. Then*

$$\int_E f d\mu = \int_X f\chi_E d\mu$$

Proof. Let ψ be a simple, non-negative function such that $\psi \leq f$ on E. Then, $\psi\chi_E$ is simple, non-negative function and $\psi\chi_E \leq f\chi_E$ on X. Hence

$$\begin{aligned}
\int_E f d\mu &= \sup\left\{\int_E \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f \text{ on } E\right\} \\
&= \sup\left\{\int_X \psi\chi_E d\mu : \psi \text{ simple, } 0 \leq \psi \leq f \text{ on } E\right\} \\
&\leq \sup\left\{\int_X \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f\chi_E \text{ on } X\right\} \\
&= \int_X f\chi_E d\mu
\end{aligned}$$

Conversely, if $\psi \leq f$ on X, then $\psi(x) = 0$ if $x \in E^c$ and hence $\psi = \psi\chi_E$ so that

$$\begin{aligned}
\int_X f\chi_E d\mu &= \sup\left\{\int_X \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f\chi_E \text{ on } X\right\} \\
&\leq \sup\left\{\int_X \psi d\mu + \int_{E^c} \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f\chi_E \text{ on } X\right\} \\
&= \sup\left\{\int_X \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f\chi_E \text{ on } X\right\} \\
&= \sup\left\{\int_X \psi\chi_E d\mu : \psi \text{ simple, } 0 \leq \psi \leq f\chi_E \text{ on } X\right\} \\
&= \sup\left\{\int_E \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f \text{ on } E\right\} = \int_E f d\mu
\end{aligned}$$

which gives us the other equality. ■

In other words, we have just proved that, for a measure space $(X, \mathfrak{A}, \mu)$, $E \in \mathfrak{A}$ and $f : X \longrightarrow \mathbb{R}$ measurable and non-negative, if f is integrable over X, then it is integrable over any E.

Lemma 77 *Let $(X, \mathfrak{A}, \mu)$ be a measure space, $E \in \mathfrak{A}$ and let $f, g : E \longrightarrow \mathbb{R}$ be measurable and non-negative functions such that $f \leq g$. Then,*

$$\int_E f d\mu \leq \int_E g d\mu$$

Proof.

$$\begin{aligned}
\int_E f d\mu &= \sup\left\{\int_E \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f \text{ on } E\right\} \\
&\leq \sup\left\{\int_E \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq g \text{ on } E\right\} \\
&= \int_E g d\mu
\end{aligned}$$

where the inequality follows because we are expanding the set over which supremum is taken. ■

Lemma 78 *Let $(X, \mathfrak{A}, \mu)$ be a measure space, $E, F \in \mathfrak{A}$ such that $E \subset F$ and let $f : E \longrightarrow \mathbb{R}$ be a measurable and non-negative function. Then,*

$$\int_E f d\mu \leq \int_F f d\mu$$

Proof. Note that $f\chi_E \leq f\chi_F$ so that

$$\int_E f d\mu = \int_X f\chi_E d\mu \leq \int_X f\chi_F d\mu = \int_F f d\mu$$

by **Lemma 77**. ■

Lemma 79 *Let* $(X, \mathfrak{A}, \mu)$ *be a measure space and let* $f : E \longrightarrow \mathbb{R}$ *be a constant, non-negative function, defined by* $f(x) = c$*. Then,* $\forall x \in E$*,*

$$\int_E f d\mu = c\mu(E)$$

Proof. $\psi \leq f$ on E implies $\psi \leq cid_E$ on E for each ψ where id_E is the identity function on E. This implies

$$\int_E \psi d\mu \leq c\mu(E)$$

by **Lemma 69**. This tells us that

$$\int_E f d\mu = \sup\left\{\int_E \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f \text{ on } E\right\} \leq c\mu(E)$$

But $\psi = cid_E$ on X is another candidate so

$$c\mu(E) \leq \sup\left\{\int_E \psi d\mu : \psi \text{ simple, } 0 \leq \psi \leq f \text{ on } E\right\} = \int_E f d\mu$$

■

3.2 Integral of Sequence of Functions

Theorem 80 (Monotone Convergence Theorem) *Let* $\{f_n(x) : n \in \mathbb{N}\}$ *be a sequence of measurable functions. Assume that* $f_n(x) \to f(x)$ *and that* f_n *is an increasing sequence (i.e.,* $\forall x$*,* $0 \leq f_n(x) \leq f_{n+1}(x)$*), then*

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$$

Proof. Note that $f_n(x) \leq f(x)$ for every n so that

$$\int_E f_n d\mu \leq \int_E f d\mu$$

Also note that

$$\int_E f_n d\mu$$

is an increasing sequence of numbers so that

$$\lim_{n\to\infty} \int_E f_n d\mu \leq \int_E f d\mu$$

For the other inequality, let φ be a simple function with $f \geq \varphi \geq 0$. By definition,

$$\int_E f d\mu = \sup\left\{\int_E \psi d\mu : \psi \text{ is simple, } f \geq \psi \geq 0\right\}$$

Then, the reverse inequality will follow if we can show that

$$\lim_{n\to\infty}\int_E f_n d\mu \geq \int_E \varphi d\mu$$

Take a number $t \in (0,1)$. Then, $E_n = \{x \in E, f_n(x) \geq t\varphi(x)\} \subset E_{n+1}$. Let

$$E = \bigcup_{i=1}^{\infty} E_i$$

If $x \in E$, then $f_n(x) \to f(x)$ and, for some large n, we have $f_n(x) \geq tf(x)$. To prove this, assume, for the sake of contradiction, that $f_n(x) < tf(x)$ for every n. Then $f(x) < tf(x)$ so that $f(x) = 0$ and, therefore, $f_n(x) < tf(x) = 0$, a contradiction to the fact that $f_n(x) \geq 0$ for all x. As an aside, by continuity of the measure, $\forall A \in \mathfrak{A}$,

$$\lim_{n\to\infty}\mu(A \cap E_n) = \mu(A \cap E)$$

Now, since φ is a simple function, it can be represented by

$$\varphi(x) = \sum_{j=1}^{N} a_j \chi_{A_j}(x)$$

and, for every j,

$$\lim_{n\to\infty}\mu(A_j \cap E_n) = \mu(A_j \cap E)$$

so that

$$\int_E f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} t\varphi d\mu = \sum_{j=1}^{N} t a_j \mu(A_j \cap E_n)$$

Passing to the limit,

$$\lim_{n\to\infty}\int_{E_n} f_n d\mu$$

is bounded, so is

$$\sum_{j=1}^{N} t a_j \mu(A_j \cap E_n)$$

Also, limit can be passed inside summation. So that

$$\lim_{n\to\infty}\int_E f_n d\mu \geq t\sum_{j=1}^{N}\lim_{n\to\infty} a_j \mu(A_j \cap E_n) = t\int_E \varphi d\mu$$

which tells us that

$$\lim_{n\to\infty}\int_E f_n d\mu \geq t\int_E \varphi d\mu$$

Now, we can let $t \to 1$. ■

Problem 81 *Let* $\{f_n : n \in \mathbb{N}\}$ *be a sequence of measurable, non-negative functions. Show that*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is a measurable function and

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

Solution 82 *Let* $k \in \mathbb{N}$. *Since the finite sum of (non-negative) measurable functions is measurable, we must have*

$$\sum_{i=1}^{k} f_i(x)$$

measurable. Now, since $f_n \geq 0$, *we must have*

$$\lim_{k\to\infty} \sup_k \sum_{i=1}^{k} f_i(x) = \lim_{k\to\infty} \sum_{i=1}^{k} f_i(x)$$

Thus,

$$\lim_{k\to\infty} \sum_{i=1}^{k} f_i(x)$$

is measurable. That is,

$$\lim_{k\to\infty} \sum_{i=1}^{k} f_i(x) = f(x)$$

is measurable. Again, since the finite sum of (nonnegative), measurable functions is measurable, we have that

$$\sum_{i=1}^{k} f_i(x)$$

is measurable. We can then have

$$\int_X \sum_{i=1}^{k} f_i(x)\, d\mu$$

Since each f_i *is non-negative, we have*

$$\int_X \sum_{i=1}^{k} f_i(x)\, d\mu = \sum_{i=1}^{k} \int f_i(x)\, d\mu$$

by linearity of the integral. Let

$$g_k(x) = \sum_{i=1}^{k} f_i(x)$$

Then, $\{g_n(x) : n \in \mathbb{N}\}$ *is a sequence of measurable functions,* $g_n(x) \to f(x)$ *and* $0 \leq g_n(x) \leq g_{n+1}(x)$. *Thus, by the Monotone Convergence Theorem,*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X g_n d\mu = \lim_{n\to\infty} \int_X \sum_{i=1}^{n} f_i(x)\, d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_X f_i(x)\, d\mu = \sum_{i=1}^{\infty} \int_X f_i(x)\, d\mu$$

Problem 83 *Show that the converse of the Monotone Convergence Theorem fails.*

Solution 84 *The following is a modification of a classic example, called Marching Intervals. For* $n \in \mathbb{N}$ *(excludes zero!), define* $f_n : [0,1] \longrightarrow \mathbb{R}$ *as follows*

$$f_{2n-1}(x) = \begin{cases} 1 & x \in \left[0, \frac{1}{2^n}\right] \\ 0 & \text{otherwise} \end{cases}$$

$$f_{2n}(x) = \begin{cases} 1 & x \in \left[\frac{1}{2^n}, 1\right] \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\lim_{n\to\infty} \int f_n d\mathfrak{m}_1 = 0$$

since the intervals are shrinking. However, $f_n \not\to 0$ *since the sequence is oscillating.*

Corollary 85 *Let* $(X, \mathfrak{A}, \mu)$ *be a measure space and let* $f, g : X \longrightarrow \mathbb{R}$ *be nonnegative, measurable functions. Then,*

$$\int_X (f+g)\, d\mu = \int_X f d\mu + \int_X g d\mu$$

It is possible to prove this corollary using the definition directly but let's use the Monotone Convergence Theorem.

Proof. Let φ_n and ψ_n be an increasing sequences of simple functions such that $\varphi_n \nearrow f$ and $\psi_n \nearrow g$. Then, $\varphi_n + \psi_n \nearrow f + g$ so that

$$\begin{aligned} \int_X (f+g)\, d\mu &= \int_X \lim_{n\to\infty} (\varphi_n + \psi_n)\, d\mu = \lim_{n\to\infty} \int_X (\varphi_n + \psi_n)\, d\mu \\ &= \lim_{n\to\infty} \int_X \varphi_n d\mu + \lim_{n\to\infty} \int_X \psi_n d\mu = \int_X f d\mu + \int_X g d\mu \end{aligned}$$

■

Corollary 86 *Let $(X, \mathfrak{A}, \mu)$ be a measure space, $A, B \in \mathfrak{A}$ and let $f : X \longrightarrow \mathbb{R}$ be a non-negative, measurable function. If $A \cap B = \varnothing$, then*

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$$

Proof.

$$\int_{A\cup B} f d\mu = \int_X f\chi_{A\cup B} d\mu = \int_X (f\chi_A + f\chi_B)\, d\mu$$

since $A \cap B = \varnothing$. Thus,

$$\int_{A\cup B} f d\mu = \int_X f\chi_A d\mu + \int_X f\chi_B d\mu = \int_A f d\mu + \int_B f d\mu$$

■

Lemma 87 (Fatou) *If $f_n \geq 0$ is measurable, then*

$$\lim_{N\to\infty} \inf_{n\geq N} \int_X f_n d\mu \geq \int_X \left(\lim_{N\to\infty} \inf_{n\geq N} f_n \right) d\mu$$

Notice that there is no mention of the limit of f.

Proof. Let

$$g_N(x) = \inf_{n\geq N} f_n(x)$$

Then, $g_N \nearrow$ and

$$\lim_{N\to\infty} g_N(x) = \lim_{N\to\infty} \inf_{n\geq N} f_n(x)$$

By the Monotone Convergence theorem,

$$\lim_{N\to\infty} \int_X g_N d\mu = \int_X \lim_{N\to\infty} g_N d\mu = \int_X \lim_{N\to\infty} \inf_{n\geq N} f_n d\mu$$

so that

$$\int_X g_N d\mu \leq \int_X f_n d\mu$$

which implies

$$\lim_{n\to\infty} \inf_{n\geq N} \int_X g_N d\mu \leq \lim_{n\to\infty} \inf_{n\geq N} \int_X f_n d\mu$$

and hence

$$\lim_{N\to\infty} \int_X g_N d\mu = \int_X \lim_{N\to\infty} g_N d\mu \leq \lim_{N\to\infty} \inf_{n\geq N} \int_X f_n d\mu$$

That is,

$$\lim_{n\to\infty} \inf_{n\geq N} \int_X f_n d\mu \geq \int_X \left(\lim_{n\to\infty} \inf_{n\geq N} f_n \right) d\mu$$

■

Exercise 88 *Give an example where equality does not hold and another where the inequality does not hold.*

Solution 89 *Let $E = \mathbb{R}$ and consider $f_n = \chi_{(n,n+1)}$ for $n \in \mathbb{N}$. Then, $f_n \longrightarrow \mathbf{0}$ pointwise, where $\mathbf{0}$ is the zero function. However,*

$$\int_E \lim_{N\to\infty} \inf_{n\geq N} f_n d\mu = \int_E (0)\, d\mu = 0 < 1 = \lim_{N\to\infty} \inf_{n\geq N} \int_E f_n d\mu$$

If we have a sequence of increasing functions $\{f_n : n \in \mathbb{N}\}$ such that $f_n \to f$, then

$$\lim_{N\to\infty} \inf_{n\geq N} \int_X f_n d\mu = \int_X f d\mu$$

Theorem 90 (Chebyshev Inequality) *Let $f \geq 0$ be a measurable function and $\lambda > 0$ be a constant. Then,*

$$\mu(X_\lambda) \leq \frac{1}{\lambda} \int_X f d\mu$$

where $X_\lambda = \{x \in X : f(x) \geq \lambda\} = f^{-1}[\lambda, \infty]$ is a measurable set.

Proof. Since $X_\lambda \subset X$, we must have

$$\int_X f d\mu \geq \int_{X_\lambda} f d\mu \geq \int_{X_\lambda} \lambda d\mu = \lambda \mu(X_\lambda)$$

■

Problem 91 *Let f be a nonnegative measurable function on X such that*

$$\int_X f d\mu = 0$$

Show that $f = 0$ almost everywhere on X.

Solution 92 *Let $\epsilon > \frac{1}{n}$ and $Y_n = \left\{x : f(x) \geq \frac{1}{n}\right\}$. Since f is measurable, we must have $Y_n \in \mathfrak{A}$ and that $\mu(Y_n) \geq 0$. Then, by Chebyshev's inequality,*

$$\mu(Y_n) \leq \frac{1}{\epsilon} \int_X f d\mu = 0$$

Thus, $\mu(Y_n) = 0$ for every $\epsilon > \frac{1}{n}$. Now let

$$Y = \bigcup_{n=1}^{\infty} Y_n$$

Then, $Y = \{x : f(x) > 0\}$ and, by countable subadditivity of μ,

$$\mu(Y) \leq \sum_{n=1}^{\infty} \mu(Y_n) = 0$$

Thus, $\mu(Y) = 0$. The result follows from Borel-Cantelli's Lemma.

Problem 93 *Show that Fatou's Lemma implies Monotone Convergence Theorem.*

Solution 94 *Let* $\{f_n(x) : n \in \mathbb{N}\}$ *be a sequence of non-negative measurable functions with* $f_n(x) \to f(x)$ *and* $0 \le f_n(x) \le f_{n+1}(x)$*. We need to show that*

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$$

We can view

$$\int_E f_n d\mu$$

as a sequence of positive, real numbers. In general, we have that

$$\lim_{n\to\infty} \sup_{k\ge n} \int_E f_k d\mu \ge \lim_{n\to\infty} \inf_{k\ge n} \int_E f_k d\mu \tag{3}$$

Since $f_n \le f_{n+1}$ *and* $f_n \to f$*, it must be that* $f_n \le f$ *for all* n*. Thus,*

$$\int_E f_n d\mu \le \int_E f d\mu$$

for all n*. Since this is valid for each* n*, we must have*

$$\sup_{k\ge n} \int_E f_k d\mu \le \int_E f d\mu$$

Letting $n \to \infty$ *on both sides gives us*

$$\lim_{n\to\infty} \sup_{k\ge n} \int_E f_k d\mu \le \int_E f d\mu \tag{4}$$

Since $f_n(x) \to f(x)$*, we must have*

$$\lim_{n\to\infty} \inf_{k\ge n} f_k = \lim_{n\to\infty} f_n = f$$

so that

$$\lim_{n\to\infty} \sup_{k\ge n} \int_E f_k d\mu \le \int_E \lim_{n\to\infty} \inf_{k\ge n} f_k d\mu$$

by ***Eq*** (4)*. This and* ***Eq*** (3) *together imply that*

$$\lim_{n\to\infty} \inf_{k\ge n} \int_E f_k d\mu = \lim_{n\to\infty} \sup_{k\ge n} \int_E f_k d\mu$$

and, therefore

$$\lim_{n\to\infty} \int_E f_n d\mu = \lim_{n\to\infty} \inf_{k\ge n} \int_E f_k d\mu = \lim_{n\to\infty} \sup_{k\ge n} \int_E f_k d\mu$$

Then, by Fatou's lemma, we have the chain

$$\lim_{n\to\infty}\inf_{k\geq n}\int_E f_k d\mu \leq \lim_{n\to\infty}\sup_{k\geq n}\int_E f_k d\mu \leq \int_E \lim_{n\to\infty}\inf_{k\geq n} f_k d\mu \leq \lim_{n\to\infty}\inf_{k\geq n}\int_E f_k d\mu \tag{5}$$

That is,

$$\lim_{n\to\infty}\sup_{k\geq n}\int_E f_k d\mu = \lim_{n\to\infty}\inf_{k\geq n}\int_E f_k d\mu = \lim_{n\to\infty}\int_E f_n d\mu \tag{6}$$

(3) *and* (6) *together imply*

$$\lim_{n\to\infty}\sup_{k\geq n}\int_E f_k d\mu \leq \int_E f d\mu = \lim_{n\to\infty}\inf_{k\geq n}\int_E f_k d\mu = \lim_{n\to\infty}\int_E f_n d\mu$$

Thus, (5) *can be re-written as*

$$\lim_{n\to\infty}\int_E f_n d\mu \leq \int_E f d\mu \leq \lim_{n\to\infty}\int_E f_n d\mu$$

so that

$$\lim_{n\to\infty}\int_E f_n d\mu = \int_E f d\mu$$

3.3 Integral as a Measure

Can we define the integral differently? No. There is, in a certain sense, uniqueness.

Theorem 95 *Let* $\mu^+(X)$ *be the set of all measurable non-negative functions* $f : X \longrightarrow \mathbb{R}$. *Assume* $J : \mu^+(X) \times \mathfrak{A} \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ *such that*

1. $J(f, A) \geq 0$ *for all* f *and* A
2. $A, B \in \mathfrak{A}$ *with* $A \cap B = \varnothing \implies J(f, A\cup B) = J(f, A) + J(f, B)$
3. $f(x) = c$ *for some constant* c *on* $A \Longrightarrow J(f, A) = cJ(id, A)$
4. $f_n \nearrow f \implies \lim_{n\to\infty} J(f_n, A) = J(f, A)$

Then, J *is unique and*

$$J(f, A) = \int_A f d\nu$$

Before we enter a proof of **Theorem 95**, we demonstrate the following. These can indeed be proven using the fact that $J(f, A)$ has an integral representation but let us show that this follows from the 4 properties above.

Corollary 96 $J(\chi_A, X) = J(id, A)$

Proof. $J(\chi_A, X) = J(\chi_A, A \cup A^c) = J(\chi_A, A) + J(\chi_A, A^c)$ by (2)
$= 1J(id, A) + 0J(id, A^c)$ by (3)
$= J(id, A)$ ■

Corollary 97 $J(f, \varnothing) = 0.$

Proof. $J(f, \varnothing) = J(f, \varnothing \cup \varnothing) = J(f, \varnothing) + J(f, \varnothing)$ by (2) so that $J(f, \varnothing) = 0$
■

Corollary 98 *Let f and g be measurable functions. Then,* $J(\alpha f + \beta g, A) = \alpha J(f, A) + \beta J(g, A)$

Proof. Let us start with simple functions:

$$\phi = \sum_{i=1}^{n} c_i \chi_{A_i} \text{ and } \varphi = \sum_{i=1}^{n} d_i \chi_{A_i}$$

where A_i is a partition of X. Then, by using induction on (2), we get

$$J(\phi + \varphi, A) = J\left(\phi + \varphi, A \cap \left(\bigcup_{i=1}^{n} A_i\right)\right) = \sum_{i=1}^{n} J(\phi + \varphi, A \cap A_i)$$

By (3), we get

$$\begin{aligned} J(\phi + \varphi, A) &= \sum_{i=1}^{n} (c_i + d_i) J(id, A \cap A_k) \\ &= \sum_{i=1}^{n} c_i J(id, A \cap A_k) + \sum_{i=1}^{n} d_i J(id, A \cap A_k) \\ &= J(\phi, A) + J(\varphi, A) \end{aligned}$$

Now since f, g are measurable, $\exists$ simple f_n, g_n such that $f_n \nearrow f$ and $g_n \nearrow g$. Then, (4) gives us the required result. Similarly, we can show that $J(\alpha f, A) = \alpha J(f, A)$. ■

And now, for a proof of **Theorem 95**.

Proof. Let $\nu(A) = J(id, A)$. We show that ν is a measure. $\nu(\varnothing) = 0$ follows immediately from **Corollary 97**. To show countable additivity, we show continuity from below. Let $A_n \subset A_{n+1}$ be an increasing sequence of measurable sets and let

$$\bigcup_{i=1}^{\infty} A_i = A$$

Then, $\chi_{A_n} \leq \chi_{A_{n+1}}$ and $\chi_{A_n} \to \chi_A$ pointwise on X. By (4), $\lim_{n\to\infty} J\left(\chi_{A_n}, X\right) = J(\chi_A, X)$. By **Corollary 96**, $\nu(A) = J(id, A) = J(\chi_A, X) = \lim_{n\to\infty} J\left(\chi_{A_n}, X\right) = \lim_{n\to\infty} J(id, A_n) = \lim_{n\to\infty} \nu(A_n)$.

Now, let

$$f = \sum_{i=1}^{n} c_i \chi_{A_i}$$

be a simple function. Then,

$$\begin{aligned}
J(f, A) &= J\left(f, \bigcup_{i=1}^{n} A \cap A_k\right) = \sum_{i=1}^{n} J(f, A \cap A_k) \text{ by } (2) \\
&= \sum_{i=1}^{n} c_i J(id, A \cap A_k) \text{ by } (3) \\
&= \sum_{i=1}^{n} c_i \nu(A \cap A_k) = \int_A f d\nu
\end{aligned}$$

Now, for an arbitrary measurable function, we can always come up with an increasing sequence of simple functions which converge to our arbitrary function. ■

What follows in this section will be a general treatment of obtaining a measure from another.

Theorem 99 *Let $(X, \mathfrak{A}, \sigma)$ be a measurable space. Fix a measurable function $f \geq 0$. Define*

$$\varphi_f(A) = \int_A f d\mu$$

Then, φ_f is a measure on $\mathfrak{A}$.

Proof. It is clear that $\varphi_f(\varnothing) = 0$. It is also easy to show that φ_f is finitely additive and monotone, by definition. To prove that φ_f is countably additive, we can show φ_f is continuous from below: that is, for any increasing sequence $A_1 \subset A_2 \subset ...$ of measurable sets,

$$\lim_{n\to\infty} \varphi_f(A_n) = \varphi_f(A)$$

where

$$A = \bigcup_{i=1}^{\infty} A_i$$

Define $g_n(x) = f(x)\chi_{A_n}(x)$. Then, g_n is an increasing sequence and converges to $f.\chi_A$, that is $g_n \nearrow f.\chi_A$. By Monotone Convergence Theorem,

$$\lim_{n\to\infty} \varphi_f(A_n) = \lim_{n\to\infty} \int_X g_n d\mu = \int_X f\chi_A d\mu = \int_A f d\mu = \varphi_f(A)$$

■

Lemma 100 (Absolute continuity) $\mu(A) = 0 \Longrightarrow \varphi_f(A) = 0$

Proof. If f is simple, we get that

$$\varphi_f(A) = \sum_{k=1}^{N} a_k \mu(A \cap A_k) \leq \mu(A) = 0$$

If f is not simple, then take $f_n \nearrow f$ with f_n simple. By the Monotone Convergence Theorem,

$$\lim_{n \to \infty} \int_A f_n d\mu = \int_A f d\mu$$

Since all the integrals on the left side are 0, we must have

$$\int_A f d\mu = 0$$

■

Lemma 101 *Assume that*

$$\int_X f d\mu < \infty$$

Then, f is finite almost everywhere. Moreover,

$$\lim_{\mu(E) \to 0} \int_E f d\mu = 0$$

Proof. We can let

$$\int_X f d\mu \leq c$$

for some constant c. Now, note that

$$A = \{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \{f(x) \geq n\}$$

so that $\mu(A) \leq \mu(A_n)$ and by Chebyshev's Inequality,

$$\mu(A_n) \leq \frac{1}{n} \int_X f d\mu \leq \frac{c}{n}$$

Since $c < \infty$, we can pass to the limit to get $\mu(A) = 0$.

For the second part, we need to prove that $\forall \epsilon > 0$, there is a $\delta > 0$ such that

$$\mu(E) < \delta \Longrightarrow \int_E f d\mu < \epsilon$$

In other words, $\mu(E) < \delta \Longrightarrow \varphi_f(E) < \epsilon$.

If f is simple, then

$$\int_X f d\mu = \sum_{k=1}^{N} a_k \mu(A_k)$$

where we can assume that $a_k < \infty$, since otherwise we have equality trivially. Then,

$$\int_E f d\mu = \sum_{k=1}^{N} a_k \mu \left(E \cap A_k\right) \leq \mu \left(E\right) \sum_{k=1}^{N} a_k$$

Thus, given $\epsilon > 0$, we can set

$$\delta = \epsilon / \sum_{k=1}^{N} a_k$$

If f is not simple, then $\exists$ a sequence of simple functions $\{g_n : n \in \mathbb{N}\}$ such that $g_n \to f$. Take a simple function $g_\epsilon < f$ from the sequence of simple functions such that

$$\int_X g_\epsilon d\mu \leq \int_X f d\mu < \int_X g_\epsilon d\mu + \frac{\epsilon}{2}$$

where g_ϵ, because it is integrable, has finitely many values and they are all finite so that g_ϵ is bounded (this is different from $1/x$, say, which does not have finitely many values). Thus, $g_\epsilon \left(x\right) \leq c_\epsilon$ for any $x \in X$ and so,

$$\int_X f d\mu = \int_X \left(f - g_\epsilon\right) d\mu + \int_X g_\epsilon d\mu \leq \int_X \left(f - g_\epsilon\right) d\mu + c_\epsilon \mu \left(X\right)$$

Also,

$$\int_X \left(f - g_\epsilon\right) d\mu < \frac{\epsilon}{2}$$

and now take $\delta = \frac{\epsilon}{2c_\epsilon}$. Then,

$$\begin{aligned}
\mu \left(E\right) &< \frac{\epsilon}{2c_\epsilon} \Longrightarrow \varphi_f \left(E\right) = \int_E f d\mu \\
&= \int_X f \chi_E d\mu < \int_X g_\epsilon \chi_E d\mu + \frac{\epsilon}{2} \\
&\leq c_\epsilon \mu \left(E\right) + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

■

3.4 Integral of Continuous Functions

Let $(X, \mathfrak{A}, \mu)$ be a measure space and let $f\left(x\right)$ be a μ-measurable function and let $g\left(x\right) = f\left(x\right)$ if $f\left(x\right) \geq 0$ and $= 0$ otherwise. Then, $g\left(x\right)$ is integrable. For the negative part, let $h\left(x\right) = -f\left(x\right)$. Then, $h\left(x\right)$ is also integrable and we define

$$\int_X f d\mu = -\int_X h d\mu$$

for $f\left(x\right) \leq 0$. In general, provided that f is finite, we can decompose $f\left(x\right)$ as $g\left(x\right) + h\left(x\right)$. There is a cleaner, more standard way of doing this: we let $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. f^+ and f^- are μ-measurable by **Corollary 49**. Moreover, both functions are non-negative and $f = f^+ - f^-$. Denote $|f| = f^+ + f^-$. If

$$\int_E f^+ d\mu, \int_E f^- d\mu < \infty$$

then we say that f is μ-summable (i.e. finite). The standard terminology is "integrable", i.e. the integral is well-defined but we will prefer summable. Thus, we can define

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$$

for any $E \in \mathfrak{A}$.

Example 102 *Let $X = N = \{1, 2, ...\}$, $\mu(A) = |A|$ (counting measure), $A \subset X$. Let $g : \mathbb{N} \longrightarrow \mathbb{R}$ be a sequence $g(n) = g_n < \infty$. Then,*

$$\int_E g d\mu = \sum_{n \in E} g_n$$

To show this, note that we can write g as

$$g = \sum_{n=1}^{\infty} c_n \chi_{\{n\}}$$

where $c_n = g(n)$ and $g_k = c_k \chi_{\{k\}}$. Note that

$$\int_{\mathbb{N}} g_k d\mu = c_k$$

Furthermore, g_k is a sequence of measurable functions such that

$$g = \sum_{n=1}^{\infty} g_n$$

*Thus, by **Problem 81,***

$$\int_{\mathbb{N}} g d\mu = \sum_{n=1}^{\infty} \int_{\mathbb{N}} g_k d\mu = \sum_{n=1}^{\infty} g(n)$$

Things are, however, not always this clean. The above representation rests on the assumption that $g(n) < \infty$ for each n. If we do not have this guarantee, we can run across $\infty - \infty$. Moreover, we have convergence issues to deal with, as well. Therefore, it is not always true that

$$\int_X f d\mu \neq \sum_{n=1}^{\infty} f_n$$

What if $f_n = \frac{(-1)^n}{n}$*? The sequence converges and so does the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

On the other hand,

$$\sum_{n=1}^{\infty} f_n^+$$

diverges. Same for f_n^-*, so*

$$\int_X f d\mu$$

is not well-defined!

Problem 103 *Let* $x \in X$ *and let* $\left(X, 2^X, \delta_x\right)$ *be a measure space where* δ_x *is defined as*

$$\delta_x (E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

If f *is a nonnegative function such that*

$$\int f d\delta_x < \infty,$$

What can we say about f*?*

Solution 104 f *is measurable since for any measurable subset of the codomain of* f*, the pre-image is a subset of* X *and hence in the* σ*-algebra – measurable. We also know that* f *is summable because the integral is finite. By* ***Lemma 101****,* f *is finite almost everywhere. That is, the set*

$$\bigcap_{n=0}^{\infty} \{y \in X : |f (y)| > n\}$$

has δ_x*-measure zero. Hence*

$$x \notin \bigcap_{n=0}^{\infty} \{y \in X : |f (y)| > n\}$$

That is, $\forall n$*,* $x \notin \{y \in X : |f (y)| \geq n\}$*. In particular,* $|f (x)| \leq 0$*. Since* x *is arbitrary, we can conclude that* f *is the zero function.*

Lemma 105 *Let* f *be a* μ*-measurable function, defined on a measure space* $(X, \mathfrak{A}, \mu)$*. Then,* f *is* μ*-summable if and only if* $|f|$ *is* μ*-summable. Moreover,*

$$\left| \int_X f d\mu \right| \leq \int_X |f| \, d\mu$$

This follows easily from the basic definition. This feature is super important and is what makes Lebesgue integral different from Riemann integral, which is discussed in the next section.

Proof. f is μ-summable $\iff$

$$\text{both } \int_E f^+ d\mu < \infty \text{ and } \int_E f^- d\mu < \infty$$

Then,

$$\int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| \, d\mu < \infty$$

and conversely. Furthermore,

$$\begin{aligned} \left| \int_E f d\mu \right| &= \left| \int_E f^+ d\mu - \int_E f^- d\mu \right| \leq \left| \int_E f^+ d\mu \right| + \left| \int_E f^- d\mu \right| \\ &= \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| \, d\mu \end{aligned}$$

■

Problem 106 *Show that the converse doesn't hold*

Solution 107 *Let $(X, \mathfrak{A}, \mu)$ be a measure space with $\mu(X) < \infty$ and let $A \subset X$ such that $A \notin \mathfrak{A}$. Define $f = -1 + 2\chi_A$. Then, $|f(x)| = 1$ for all x so that the $|f|$ is summable. However, f is not measurable since A is not measurable.*

Problem 108 *Let f_n be a sequence of summable functions and $f_n \longrightarrow f$ for a summable f. Show that*

$$\int_X |f_n - f| \, d\mu \to 0 \iff \int_X |f_n| \, d\mu \to \int_X |f| \, d\mu$$

Solution 109 *Let $f_n = f_n^+ - f_n^-$ and $f = f^+ - f^-$, and $f_n - f = g_n = g_n^+ - g_n^-$. Rearranging gives us $g_n^+ + f_n^- + f^+ = f_n^+ + f^- + g_n^-$ so that*

$$\int_X g_n^+ d\mu + \int_X f_n^- d\mu + \int_X f^+ d\mu = \int_X f_n^+ d\mu + \int_X f^- d\mu + \int_X g_n^- d\mu \qquad (7)$$

Now, the first condition is equivalent to

$$\begin{aligned} \int_X |f_n - f| \, d\mu &\to 0 \iff \int_X |g_n| \, d\mu \to 0 \iff \int_X g_n^+ d\mu + \int_X g_n^- d\mu \to 0 \\ &\iff \int_X g_n^+ d\mu \to 0 \text{ and} \int_X g_n^- d\mu \to 0 \end{aligned}$$

so that applying limit to ***Eq*** (7) *gives us*

$$\begin{aligned}
\lim_{n\to\infty}\int_X f_n^- d\mu + \int_X f^+ d\mu &= \lim_{n\to\infty}\int_X f_n^+ d\mu + \int_X f^- d\mu \\
&\iff \lim_{n\to\infty}\left(\int_X f_n^+ d\mu - \int_X f_n^- d\mu\right) \\
&= \lim_{n\to\infty}\int_X f_n d\mu = \int_X f^+ d\mu - \int_X f^- d\mu = \int_X f d\mu \\
&\iff \int_X f_n d\mu \to \int_X f d\mu \iff \int_X |f_n|\, d\mu \to \int_X |f|\, d\mu
\end{aligned}$$

Now for some notation: if f is μ-summable, then write $f \in \mathcal{L}^1(\mu)$. The 1 in the superscript is immaterial for now but its importance will become relevant when we cover L^p spaces.

Theorem 110 *Let $f, g \in \mathcal{L}^1(\mu)$ and $t \in \mathbb{R}$. Then, $tf \in \mathcal{L}^1(\mu)$, $f + g \in \mathcal{L}^1(\mu)$ and*

$$\int_E tf d\mu = t\int_E f d\mu \text{ and } \int_E (f+g)\, d\mu = \int_E f d\mu + \int_E g d\mu$$

Proof. Assume that $t = 0$. Then, $\int_E tf d\mu = 0$ so that $tf \in \mathcal{L}^1(\mu)$. If $t \neq 0$, we know that tf is measurable by **Corollary 49**. We show that $|tf|$ is summable.

$$\int_E |tf|\, d\mu = \int_E |t|\,|f|\, d\mu = \int_E \left(|t|\, f^+ + |t|\, f^-\right) d\mu = \int_E |t|\, f^+ d\mu + \int_E |t|\, f^- d\mu < \infty$$

since the latter two are finite, because f^+ and f^- are both summable.

Next, let $h = f + g$ so that $h^+ - h^- = f^+ - f^- + g^+ - g^-$. Then, $h^+ + f^- + g^- = f^+ + g^+ + h^-$ so we can use linearity since we are working with non-negative functions. This gives us

$$\int_E h^+ d\mu + \int_E f^- d\mu + \int_E g^- d\mu = \int_E f^+ d\mu + \int_E g^+ d\mu + \int_E h^- d\mu$$

Rearranging this gives us

$$\begin{aligned}
\int_E h^+ d\mu - \int_E h^- d\mu &= \int_E f^+ d\mu - \int_E f^- d\mu + \int_E g^+ d\mu - \int_E g^- d\mu \\
&= \int_E f d\mu + \int_E g d\mu
\end{aligned}$$

We know that h is measurable but is it summable? As shown in **Problem 106**, it does not follow that $h \notin \mathcal{L}$. Note that since the two terms on the RHS are finite, it follows that the term on the LHS is finite.

$$\begin{aligned}
\left|\int_E h^+ d\mu - \int_E h^- d\mu\right| &\leq \left|\int_E h^+ d\mu\right| + \left|\int_E h^- d\mu\right| \\
&\leq \int_E \left|h^+\right| d\mu + \int_E \left|h^-\right| d\mu = \int_E h^+ d\mu + \int_E h^- d\mu \\
&\Longrightarrow \left|\int_E h^+ d\mu - \int_E h^- d\mu\right| \leq \int_E h^+ d\mu + \int_E h^- d\mu \\
&\Longrightarrow \left|\int_E h d\mu\right| \leq \int_E f d\mu + \int_E g d\mu
\end{aligned}$$

because $h^+ = f^+ + g^+$ and $h^- = f^- + g^-$. Therefore, $h = f + g \in \mathcal{L}^1(\mu)$ and the Lebesgue Integral is linear. ■

We are assuming that

$$\int_E f^{\pm} d\mu < \infty \text{ and } \int_E g^{\pm} d\mu < \infty$$

so, by **Lemma 101**, the four functions $f^{\pm}$ and $g^{\pm}$ are finite almost everywhere. Therefore, $f + g$ is well defined and finite on $X \backslash N$ where $\mu(N) = 0$ and we can extend $f + g$ to be (honestly) finite by defining $(f + g)(n) = 0$ for $n \in N$. The product of two summable functions is not summable, however. For a simple example, take $I = (0, 1]$ and $f(x) = g(x) = 1/\sqrt{x}$ and $\mu = \mathfrak{m}_1$.

Theorem 111 *Let $f, g \in \mathcal{L}^1(\mu)$ with $f \leq g$. Then, for any $E \in \mathfrak{A}$,*

$$\int_E f d\mu \leq \int_E g d\mu.$$

Proof. Again, $f^+ - f^- \leq g^+ - g^-$ so that $f^+ + g^- \leq f^- + g^+$. We can then use linearity of the integral. ■

Theorem 112 (Lebesgue Dominated Convergence Theorem) *Let $f_n \in \mathcal{L}^1(\mu)$, $f_n(x) \to f(x)$ for every x and $|f_n(x)| \leq F(x)$ for every x and n for some $F \in \mathcal{L}^1(\mu)$. Then,*

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu.$$

Proof. Case 1. $f(x) = 0$ for all $x \in X$ and $f_n \geq 0$.

Then,

$$\int_E f_n d\mu \geq 0$$

so that

$$\lim_{n\to\infty} \int_E f_n d\mu = \lim_{n\to\infty} \inf_{n \geq N} \int_E f_n d\mu \geq 0.$$

Let us consider $F(x) - f_n(x) \geq 0$. By Fatou's lemma,

$$\lim_{n\to\infty} \inf_{n\geq N} \int_E (F - f_n)\, d\mu \geq \int_E \lim_{n\to\infty} \inf_{n\geq N} (F - f_n)\, d\mu.$$

The lower limit is not linear! It is monotone in one direction but not linear. Recall that

$$\lim_{n\to\infty} \inf_{n\geq N} (A + a_n) = A + \lim_{n\to\infty} \inf_{n\geq N} a_n \text{ and } \lim_{n\to\infty} \inf_{n\geq N} (-a_n) = -\lim_{n\to\infty} \sup_{n\geq N} a_n$$

Thus,

$$\lim_{n\to\infty} \inf_{n\geq N} \int_E (F - f_n)\, d\mu \geq \int_E \left(F - \lim_{n\to\infty} \sup_{n\geq N} f_n \right) d\mu$$

Since $f_n \to 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} \inf_{n\geq N} \int_E (F - f_n)\, d\mu &= \lim_{n\to\infty} \inf_{n\geq N} \left(\int_E F d\mu - \int_E f_n d\mu \right) \geq \int_E F d\mu \\
&\implies \int_E F d\mu - \lim_{n\to\infty} \sup_{n\geq N} \int_E f_n d\mu \geq \int_E F d\mu \\
&\implies \lim_{n\to\infty} \sup_{n\geq N} \int_E f_n d\mu \leq 0 \\
&\implies \lim_{n\to\infty} \int_E f_n d\mu = 0 = \int_E f d\mu
\end{aligned}$$

The rest of the cases are left as an exercise. ■

Lemma 113 *Let $(X, \mathfrak{A}, \mu)$ be a measurable space with $\mu(X) < \infty$. Let f be a summable function. Then, f is finite almost everywhere.*

Proof. Consider $f_n = n\chi_{E_n}$ and let $E_n = \{x \in X : f(x) \geq n\}$. Then, we have a decreasing sequence of sets $E_n \subset E_{n-1}$. Now for all x in E_n we have that $n\chi_{E_n}(x) = n \leq |f(x)|$. From the monotonicity of the integral,

$$n\mu(E_n) = \int_X n\chi_{E_n} d\mu \leq \int_{E_n} |f| d\mu \leq \int_X |f| d\mu = C < \infty$$

for some $C \in \mathbb{R}$. That is, $\mu(E_n) \leq C/n$. Now set

$$E = \bigcap_{n=1}^{\infty} E_n$$

i.e., x belongs to E iff $|f(x)| = \infty$. Since $\mu(E_1) < \infty$ (because $\mu(X) < \infty$), then by continuity of measure,

$$0 \leq \mu(E) \leq \mu(E_n) \leq C/n \implies \mu(E) = 0$$

■

Corollary 114 *Let* $(X, \mathfrak{A}, \mu)$ *be a measurable space with* $\mu(X) < \infty$ *and let* f_n,f *be functions which are finite almost everywhere for each* n. *Assume that* $|f_n| \leq C$ *for each* n *and* $f_n \longrightarrow f$. *Then*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. We can let $C(x) = C$ be a constant function. Then $C \in \mathcal{L}^1(\mu)$. Moreover, note that, for each n,

$$\left| \int_X f_n d\mu \right| \leq \int_X |f_n| \, d\mu \leq C\mu(X) < \infty$$

hence $f_n \in \mathcal{L}^1(\mu)$. Hence by Lebesgue's Dominated Convergence Theorem, the result follows. ■

Problem 115 *Assume* $\{f_n : n \in \mathbb{N}\}$ *is a decreasing sequence of positive measurable functions with* $f_n(x) \to f(x)$ *for every* $x \in X$. *If* f_1 *is summable, show that*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

but this might not be true if f_1 *is not summable.*

Solution 116 *Since we have* $f_1 \geq f_n$ *for all* n, *and in particular,* $f_1 \geq f$. *Moreover,* $\int_X f_n d\mu \leq \int_X f_1 d\mu$. *By Fatou's Lemma,*

$$\infty > \int_X f_1 d\mu \geq \lim_{N\to\infty} \inf_{n\geq N} \int_X f_n d\mu \geq \int_X \left(\lim_{N\to\infty} \inf_{n\geq N} f_n \right) d\mu = \int_X f d\mu$$

Hence f *is summable and positive almost everywhere. The result then follows by Lebesgue's Dominated Convergence.*

Consider $\left\{ f_n = n\chi_{(0,1/n-1)} : n \in \mathbb{N} \right\}$. *Then,* f_1 *is not summable,* $f_{n+1} \leq f_n$ *and* $f_n \to \mathbf{0}$ *pointwise but*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \lim_{n\to\infty} \left(\frac{n}{n-1} \right) = 1 \neq \int_{\mathbb{R}} f d\mu = 0$$

Problem 117 *Suppose* μ *is a positive measure on* X, f *is a non-negative measurable function and*

$$\int_X f d\mu = c < \infty$$

Show that

$$\lim_{n\to\infty} \int_X n \log\left(1 + (f/n)^\alpha\right) d\mu = \begin{cases} \infty & \text{for } \alpha \in (0,1) \\ c & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}$$

Solution 118 *Let $f_n(x) = n \log(1 + (f(x)/n)^\alpha)$. Then, $f_n \geq 0$. Consider the case that $0 < \alpha < 1$. Then, $(f(x)/n)^\alpha \to 0$. By L'Hospital's Rule,*

$$\lim_{n\to\infty} \frac{\log(1 + (f(x)/n)^\alpha)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{-\alpha f(x)^\alpha / n^{\alpha+1}}{1+(f(x)/n)^\alpha}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\alpha f(x)^\alpha}{n^{-2} + f(x)^\alpha n^{-2-\alpha}} = \infty$$

Hence by Fatou's lemma, $\underline{\lim} \int_X f_n d\mu \geq \int_X (\underline{\lim} f_n)\, d\mu = \infty$ so that $\underline{\lim} \int_X f_n d\mu = \lim \int_X f_n d\mu = \infty$. Next, suppose that $\alpha = 1$ and recall that $x \geq 0 \implies \log(1+x) \leq x$. Thus,

$$f_n = \log\left(1 + \frac{f}{n}\right) \leq \frac{f}{n}$$

and so $f_n \leq f$. Thus, each f_n is summable and by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu = c$$

Finally, consider $\alpha > 1$. Then,

$$\lim_{n\to\infty} \frac{\log(1 + (f(x)/n)^\alpha)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{-\alpha f(x)^\alpha / n^{\alpha+1}}{1+(f(x)/n)^\alpha}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\alpha f(x)^\alpha}{n^{\alpha-1} + f(x)^\alpha / n} = 0$$

By Fatou's Lemma, the result follows.

3.5 Convergence in Measure

Definition 119 *Let $\mathcal{L}^0(\mu)$ be the set of μ-measurable functions which are finite almost everywhere. If $\{f_n : n \in \mathbb{N}\} \cup \{f\} \subset \mathcal{L}^0(\mu)$, we say that f_n converges to f* ***in measure****, written as $f_n \implies f$, if $\forall \epsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$.*

Problem 120 *If $\{f_n : n \in \mathbb{N}\} \cup \{f, g\} \subset \mathcal{L}^0(\mu)$, and $f_n \implies f$ and $f_n \implies g$, then $f = g$ almost everywhere*

Solution 121 *Let $\epsilon > 0$, $E_{f-g} := \{x : |f(x) - g(x)| \geq \epsilon\}$, $E_f = \left\{x : |f(x) - f_n(x)| \geq \frac{\epsilon}{2}\right\}$ and $E_g = \left\{x : |g(x) - f_n(x)| \geq \frac{\epsilon}{2}\right\}$. Then, note that $E_{f-g} \subset E_f \cup E_g$. To show this, let $x \notin E_g \cap E_f$. Then, $|f_n(x) - f(x)| < \epsilon/2$ and $|f_n(x) - g(x)| < \epsilon/2$ so that $|f(x) - g(x)| = |f(x) - f_n(x) + f_n(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)| < \epsilon$ so that $x \notin E_{f-g}$. Then, by monotonicity of μ, it follows that $\mu(E_{f-g}) \leq \mu(E_f) + \mu(E_g)$. Since $\mu(E_f), \mu(E_g) \to 0$, it follows that $\mu(E_{f-g}) \to 0$ so that $g(x) = f(x)$ almost everywhere.*

Theorem 122 (Lebesgue) *Let $(X, \mathfrak{A}, \mu)$ be a measurable space with $\mu(X) < \infty$, $\{f_n : n \in \mathbb{N}\} \cup \{f\} \subset \mathcal{L}^0(\mu)$ such that $f_n \to f$ almost everywhere. Then $f_n \implies f$.*

That is, on a set of finite measure, almost every convergence implies convergence in measure.

Proof. We will prove the theorem for the case $f_n \to f$ everywhere where f_n, f are finite everywhere. The result will then follow directly from this fact as "everywhere $\Longrightarrow$ almost everywhere".

Consider the error set $\mathcal{E}_k(\epsilon) = \{x : |f_k - f| \geq \epsilon\}$. Our goal is to show that $\mu(E_n(\epsilon)) \to 0$ where

$$E_n(\epsilon) = \bigcup_{k=n}^{\infty} \mathcal{E}_k(\epsilon)$$

Defining $E_n(\epsilon)$ this way gives us a decreasing sequence $E_1(\epsilon) \supset E_2(\epsilon) \supset ....$ Since we have convergence of f_n, we have $\mu(E_1(\epsilon)) < \infty$ so that

$$\mu(E_n(\epsilon)) \to \mu\left(\bigcap_{n=1}^{\infty} E_n(\epsilon)\right).$$

by continuity of measure. We now claim that

$$\bigcap_{n=1}^{\infty} E_n(\epsilon)$$

is empty: if

$$x \in \bigcap_{n=1}^{\infty} E_n(\epsilon)$$

then for every n, $\exists N \geq n$ such that $|f_n(x) - f(x)| \geq \epsilon$. Thus, f_n does not converge, a contradiction. Thus, $\mu(\mathcal{E}_n(\epsilon)) \to 0$. Since $0 \leq \mu(\mathcal{E}_n(\epsilon)) \leq \mu(E_n(\epsilon))$, it follows the $f_n \implies f$. ■

Converse is not true!

Example 123 *Consider $X = [0,1]$, the Lebesgue Measure and sequence $f_n = \chi_{\left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]}$ where $k = \lfloor \log_2 n \rfloor$ and $j = n - 2^k$. This sequence is called the Type Writer sequence. The first five terms of the sequence are $\chi_{\left[0,\frac{1}{2}\right]}$, $\chi_{\left[\frac{1}{2},1\right]}$, $\chi_{[0,1/4]}$, $\chi_{\left[\frac{1}{4},\frac{1}{2}\right]}$. As n increases, the intervals shrink further. Thus, for $\epsilon > 0$, the measure of the set $E_n = \{x : |f_n(x)| \geq \epsilon\}$ approaches zero so that $f_n \implies \mathbf{0}$, the zero function, so we have convergence in measure. Moreover, each f_n is finite, hence is finite almost everywhere. However, for any $x \in [0,1]$, f_n does not converge to any function since the sequence is oscillating. Hence f_n does not converge anywhere.*

The finiteness condition is necessary, as well. Consider $f_n = \chi_{(n,n+1)}$ so that $f_n \longrightarrow f \equiv 0$ almost everywhere. However, for $\epsilon = 1/n$, and $n > 1$, $\mathfrak{m}_1(E_n) = \mathfrak{m}_1(\{x : |f_n(x)| \geq 1/n\}) = \mathfrak{m}_1(n, n+1) = 1$ and this does not converge to zero.

An even stronger statement to Lebesgue is the following:

Theorem 124 (Reisz) *Every sequence that converges in measure contains a subsequence which converges almost everywhere to the same limit.*

Theorem 125 *Let* $(X, \mathfrak{A}, \mu)$ *be a measurable space,* $\{f_n : n \in \mathbb{N}\} \cup \{f\}, \{F\} \subset \mathcal{L}^1(\mu)$ *such* $f_n \implies f$ *and* $|f_n| \leq F$*. Then*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

This is not Lebesgue's Dominated Convergence Theorem since we have convergence in measure, not the usual point-wise convergence. That is, for a bounded sequence which converges in measure, the integrals coverge.

Proof. We first claim that that $|f| \leq F$ almost everywhere

To see this, we note that $|f_n| \leq F$ for all n tells us that

$$\sup_{n\geq N} |f_n| \leq F \implies \lim_{N\to\infty} \sup_{n\geq N} |f_n| = |f| \leq F$$

Case I

Assume that $\mu(X) < \infty$.

Since $|f| \leq F$, we can assert, by triangle inequality, that $|f_n - f| \leq 2F$ is valid for all x. Now let $\mathcal{E}_n(\epsilon) = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$. Such a collection is measurable. Then,

$$\int_X |f_n - f| \, d\mu = \int_{\mathcal{E}_n(\epsilon)} |f_n - f| \, d\mu + \int_{X \setminus \mathcal{E}_n(\epsilon)} |f_n - f| \, d\mu$$

The first sum is sort of a global estimate (i.e. $\leq 2F$) whereas the second sum is a local one (i.e. $< \epsilon$). Since $X \setminus \mathcal{E}_n(\epsilon) = \{x \in X : |f_n(x) - f(x)| < \epsilon\}$, we must have

$$\int_X |f_n - f| \, d\mu \leq \int_{\mathcal{E}_n(\epsilon)} 2F d\mu + \epsilon\mu(X).$$

For a fixed ϵ, by **Lemma 101**, $\mu(\mathcal{E}_n(\epsilon)) \to 0$ tells us that

$$\int_X |f_n - f| \, d\mu \leq \epsilon\mu(X).$$

We do not know if the limits exist, yet we can apply lim sup on both sides to get.

$$\lim_{n\to\infty} \sup_{k\geq n} \int_X |f_n - f| \, d\mu \leq \epsilon\mu(X).$$

Notice that in the equality, nothing depends on n on the right side and everything on the left is true for every ϵ. Thus, we can let $\epsilon \to 0$ to get

$$\lim_{n\to\infty} \sup_{k\geq n} \int_X |f_k(x) - f(x)| \, d\mu = 0$$

Now,

$$\begin{aligned} 0 &\leq \int_X |f_n(x) - f(x)| \, d\mu \\ &\implies 0 \leq \lim_{n\to\infty} \inf_{k\geq n} \int_X |f_n(x) - f(x)| \, d\mu \leq \lim_{n\to\infty} \sup_{k\geq n} \int_X |f_k(x) - f(x)| \, d\mu = 0 \end{aligned}$$

so that

$$\lim_{n\to\infty}\int_X |f_n(x) - f(x)|\, d\mu = 0$$

Since

$$\left|\int_X (f_n(x) - f(x))\, d\mu\right| \leq \int_X |f_n(x) - f(x)|\, d\mu$$

we can apply limit on both sides to conclude that

$$\lim_{n\to\infty}\int_X f_n(x)\, d\mu = \int_X f(x)\, d\mu$$

Case II

Now assume that $\mu(X) = \infty$. It can be shown that $\forall g \in \mathcal{L}^1(\mu)$ with $g \geq 0$, $\exists A \in \mathfrak{A}$ such that $\mu(A) < \infty$ and $\int_{X\backslash A} g d\mu < \epsilon$. We use this without proof.

Then, $|f_n - f| \geq 0$ and $|f_n - f| \in \mathcal{L}^1(\mu)$. Use this and the continuity of $\varphi_f(.)$ to prove the second case. The plan would be to use $\int_X |f_n - f|\, d\mu = \int_A |f_n - f|\, d\mu + \int_{X\backslash A} |f_n - f|\, d\mu$. Thus, $\int_{X\backslash A} |f_n - f|\, d\mu \leq \int_{X\backslash A} 2F d\mu$. Now use $g = 2F$. ■

We cannot hammer uniform continuity in here because $\mu(X\backslash A) = \infty$.

The assumption of boundedness of the sequence is essential, even if the space has finite measure.

Example 126 *Consider $X = (0,1)$, $f_n = n\chi_{(0,1/n)}$. We have convergence in measure to the zero function. However,*

$$\int_X f_n d\mu = n\mu(0, 1/n) = 1$$

for all n so that we do not have convergence of integrals.

Problem 127 *Let $(X, \mathfrak{A}, \mu)$ be a measurable space, with $\mu(X) < \infty$. Show that the sequence $\{f_n : n \in \mathbb{N}\}$ converges in measure to f if and only if*

$$\lim_{n\to\infty}\int_X \frac{|f_n - f|}{1 + |f_n - f|} d\mu = 0$$

Solution 128 *Let $\epsilon > 0$ and $E_n = \{x : |f_n(x) - f(x)| \geq \epsilon\}$. Since $X = E_n \cup E_n^c$, we have*

$$\int_X \frac{|f_n - f|}{1 + |f_n - f|} d\mu = \int_E \frac{|f_n - f|}{1 + |f_n - f|} d\mu + \int_{E^c} \frac{|f_n - f|}{1 + |f_n - f|} d\mu$$

By second part of ***Lemma 101****, the first limit of the first integral on the right is zero. For the second integral, we have $E_n^c = \{x : |f_n(x) - f(x)| < \epsilon\}$. Note that $\frac{|f_n - f|}{1 + |f_n - f|} < |f_n - f| < \epsilon$ so that*

$$\left|\int_{E_n^c} \frac{|f_n - f|}{1 + |f_n - f|} d\mu\right| \leq \int_{E_n^c} \left|\frac{|f_n - f|}{1 + |f_n - f|}\right| d\mu < \int_{E_n^c} \epsilon d\mu = \epsilon\mu(E_n^c) < \epsilon\mu(X)$$

Thus, in the limit, this integal is zero, as well.

Conversely, assume that the sequence does not converge in measure. That is, $\exists \epsilon > 0$ such that $\lim_{n\to\infty} \mu(E_n) \neq 0$ where $E_n = \{x : |f(x) - f_n(x)| \geq \epsilon\}$. That is, $\exists \delta > 0$ such that $\forall N$, $|\mu(E_n)| \geq \delta$ for some $n < N$. Since $E_n \subset X$, we have

$$\int_X \frac{|f_n - f|}{1 + |f_n - f|} d\mu \geq \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \geq \int_{E_n} \frac{\epsilon}{1+\epsilon} d\mu = \frac{\epsilon}{1+\epsilon} \mu(E_n) \geq \frac{\epsilon}{1+\epsilon} \delta$$

so that the sequence of integrals does not converge, a contradiction.

3.6 Riemann Integral

Definition 129 *The Riemann integral of a function f is defined as the number*

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum f(\xi_{k,n})(x_{k,n} - x_{k-1,n})$$

where $a = x_{1,n} < x_{2,n} < ... < x_{n,n} = b$ and $\xi_{k,n} \in [x_{k-1,n}, x_{k,n}]$

The first condition is that size of every partitions tends to zero and that the limit does not depend on $\xi_{k,n}$. A very important example is as follows

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

in which case the integral does depend on $\xi_{k,n}$.

If f is continuous, then the integral exists. The axioms of Lebesgue measure can be given as follows: define

$$\Psi_f(\Delta) = \int_a^b f(x)\, dx$$

where Δ is a compact interval. Then, the following hold

1. $\Psi_f(\Delta_1) + \Psi_f(\Delta_2) = \Psi_f(\Delta)$ where $\Delta_1 = [a, c]$ and $\Delta_2 = [c, b]$. This is where the trouble comes in (the end points of intervals should match).
2. $\Psi_c(\Delta) = c(b - a)$ where c is a constant function taking value c everywhere.
3. If $f \leq g$, then $\Psi_f(\Delta) \leq \Psi_g(\Delta)$

These three properties uniquely define the Riemann integral. To show the integral exists (assuming that f is continuous) on closed intervals, by uniform continuity, we can have a δ that does not depend on x. We can then split the closed interval $[a, b]$ by δ to get $f(\xi) - \epsilon \leq f(x) \leq f(\xi) + \epsilon$, which leads to the definition of the Riemann integral

Thus, instead of using the definition of Riemann integral, we could use the three axioms. These three already hold for the Lebesgue integral, thus the mantra "Riemann Integral $\implies$ Lebesgue Integral" for a compact domain.

Theorem 130 *Let $f \in \mathcal{C}([a,b])$. Then, $f \in \mathcal{L}^1(\mathfrak{m}_1)$ and*

$$\int_{[a.b]} f d\mathfrak{m}_1 = \int_a^b f(x)\,dx$$

Proof. $f^{-1}(a,\infty)$, by continuity, is open and every open set is measurable. That is, $f^{-1}(a,\infty) \in \mathfrak{M}_1$. Now, $|f(x)| \le M$ for every $x \in [a,b]$ and $\mathfrak{m}_1([a,b]) = b-a$. Note that adding or subtracting c from an interval (to make the two intervals disjoint) doesn't make much a difference since it is just erasing or adding a set of measure 0. Thus, $f \in \mathcal{L}^1(\mathfrak{m}_1)$. This essentially means

$$\varphi_f(\Delta) = \int_\Delta f d\mathfrak{m}_1.$$

By uniqueness, $\varphi_f(\Delta) = \Psi_f(\Delta)$. ■

The converse, of course fails. Recall the Dirichlet criterion for improper integrals: if the Riemann integral of a function f is uniformly bounded over all intervals, and g is a monotonically decreasing non-negative function, then the Riemann integral of fg is a convergent improper integral. Therefore, the Riemann integral is defined in

$$\int_1^\infty \frac{\sin x}{x}dx.$$

if we take $g(x) = 1/x$ and $f(x) = \sin x$. However, it has no chance of being summable because

$$\int_{(1,\infty)} \left|\frac{\sin x}{x}\right| d\mathfrak{m}_1 = \infty.$$

We will prove rigorously the above when we discuss functions of bounded variation, but for now, let us take that on faith.

There are ways to work around this limitation:

Theorem 131 *Let $f \in \mathcal{C}([a,b))$ where $b \in \mathbb{R} \cup \{\infty\}$. Then, $f \in \mathcal{L}^1(\mathfrak{m}_1) \iff$*

$$\int_a^b |f(x)|\,dx = \lim_{t\to b} \int_a^b |f(x)|\,dx < \infty$$

In this case,

$$\int_a^b |f(x)|\,dx = \int_{[a,b)} f d\mathfrak{m}_1$$

Proof. ($\Longrightarrow$)

$f \in \mathcal{L}^1(\mathfrak{m}_1) \Longrightarrow |f| \in \mathcal{L}^1(\mathfrak{m}_1)$. If $t < b$, then $f \in \mathcal{C}([a,t])$. By the previous result,

$$\begin{aligned}\int_a^t |f(x)|\,dx &= \int_{[a,t]} |f|\,d\mathfrak{m}_1 \\ &= \int_{[a,b)} |f|\,d\mathfrak{m}_1 - \int_{(t,b)} |f|\,d\mathfrak{m}_1\end{aligned}$$

Our goal is to show that

$$\lim_{t\to b}\int_{(t,b)}|f|\,d\mathfrak{m}_1=0.$$

If $b\in\mathbb{R}$, then it follows trivially since $\mathfrak{m}_1((t,b))\to 0$. If $b=\infty$, consider a monotonically increasing, unbounded sequence $\{a_n : n\in\mathbb{N}\}$ with $a_1=t$. Then,

$$\lim_{t\to b}t=\lim_{n\to\infty}a_n$$

We can then consider the decreasing sequence of intervals $(a_n,b)\supset(a_{n+1},b)$. Let $E_k=\bigcap_{n=k}^{\infty}(a_n,b)$. Then, $E_k\subset E_{k+1}$ so that by Absolute Continuity of the Lebesgue measure,

$$\lim_{t\to b}\int_{(t,b)}|f|\,d\mathfrak{m}_1=\varphi_{|f|}\left(\bigcap_{n=1}^{\infty}(a_n,b)\right)=\lim_{n\to\infty}\varphi_{|f|}(E_n)=0$$

($\Longleftarrow$) If the Riemann integral exists, then we need to prove that the Lebesgue integral exists, provided that

$$\int_a^b|f(x)|\,dx=\lim_{t\to b}\int_a^b|f(x)|\,dx<\infty$$

If $t<b$, by **Theorem 130**,

$$\varphi_{|f|}([a,t])=\int_{[a,t]}|f|\,d\mathfrak{m}_1=\int_a^t|f(x)|\,dx$$

Choose $\{t_n : n\in\mathbb{N}\}$ such that $t_n\nearrow b$. Then, $[a,t_1]\subset[a,t_2]\subset\ldots$. Since $\varphi_{|f|}$ is a measure, then, by continuity of measure,

$$\int_a^b|f(x)|\,dx=\lim_{t\to b}\int_a^t|f(x)|\,dx=\lim_{t\to b}\int_{[a,t]}|f|\,d\mathfrak{m}_1=\lim_{n\to\infty}\varphi_{|f|}([a,t_n])<\infty.$$

■

Problem 132 *Let $f(x)=x^{-1/2}$ for $x\in(0,1]$ with $f(0)=1$. Without referring to Riemann integration, prove that f is summable with respect to $\mathfrak{m}_1$*

Solution 133 *Let*

$$\varphi=\sum_{i=1}^{n}c_i\chi_{A_i}\le f$$

be a simple function with A_i being the partition of $(0,1]$. Then,

$$\int_{(0,1]}\phi d\mathfrak{m}_1=\sum_{i=1}^{n}c_i\mathfrak{m}_1(A_i\cap(0,1])<\infty$$

and so

$$\sup\left\{\int_E\psi d\mu : \psi \text{ simple, } 0\le\psi\le f \text{ on } E\right\}=\int_E fd\mu<\infty$$

3.7 Product Measures

The goal of this section is to define product of measures, and, therefore, multiple integrals. That is, $dA = dxdy = d\mathfrak{m}_1 d\mathfrak{m}_1 = d\mathfrak{m}_2$ where the product commutes. This is necessary since not in all cases do we get an iterated integral. Proving this is the main motivation, so we can ultimately compute

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

Some notation: we will have two measure spaces $(X, \mathfrak{A}_X, \mu_X)$ and $(Y, \mathfrak{A}_Y, \mu_Y)$ and define $\mathcal{P} = \{A \times B : A \in \mathfrak{A}_X \text{ and } B \in \mathfrak{A}_Y\}$. This is at least a semiring but, sadly, at most a semiring. The fact that $\mathcal{P}$ is a semiring follows from **Problem 26**. To show that the product of two measures is not a measure, even if the measure is complete, consider $X = [0,1] = Y$ and let $\mu_X = \mu_Y = \mathfrak{m}_1$. Let A be a nonmeasurable susbet of $[0,1]$. Then, $A \times \{0\} \notin \mathcal{P}$. Note that $A \times \{0\} \subset [0,1] \times \{0\}$ and the latter has measure zero. Now take $E = \left[0, \frac{1}{4}\right]$ and $F = \left[\frac{3}{4}, 1\right]$. Then, clearly $E \cup F \in \mathfrak{A}_X = \mathfrak{A}_Y$. Moreover, $E \times E, F \times F \in \mathcal{P}$. However, $(E \times E) \cup (F \times F) \notin \mathcal{P}$.

Since we've seen that we can construct a measure using an ordinary set function by Lebesgue-Carathéodory theorem, it would not be outlandish to propose a set function $\mu_0 (A \times B) = \mu_X (A) .\mu_Y (B)$, which is naturally well-defined on $\mathcal{P}$. Let us prove that it satisfies the hypothesis of the Lebesgue-Carathéodory theorem.

Lemma 134 *μ_0 is a pre-measure.*

Proof. P1 $\mu_0 (\varnothing) = \mu_X (A) .\mu_Y (\varnothing)$ or $\mu_X (\varnothing) .\mu_Y (B)$ for any $A \in \mathfrak{A}_X$ or $B \in \mathfrak{A}_Y$, and both are zero.

P2 To show that μ_0 is finitely additive, let $\{A_j : 1 \leq j \leq n\} \subset \mathfrak{A}_X$ and $\{B_k : 1 \leq k \leq m\} \subset \mathfrak{A}_Y$ be collections of disjoint sets and let

$$A = \bigcup_{j=1}^{n} A_j \text{ and } B = \bigcup_{k=1}^{m} B_k$$

Then,

$$\mu_0 (A \times B) = \mu_X (A) \mu_Y (B) = \left(\sum_{i=1}^{n} \mu (A_i) \right) \left(\sum_{k=1}^{m} \mu (B_k) \right) = \sum_{i=1}^{n} \sum_{k=1}^{m} \mu (A_i) \mu (B_k)$$

On the other hand, since

$$A \times B = \left(\bigcup_{j=1}^{n} A_j \right) \times \left(\bigcup_{k=1}^{m} B_k \right) = \bigcup_{(j,k)}^{(n,m)} (A_i \times B_k)$$

we therefore have equality in both sides.

P3 For countable monotonicity, let

$$E = A \times B \subset \bigcup_{j=1}^{\infty} A_j \times B_j = \bigcup_{j=1}^{\infty} E_j$$

where $E_j = A_j \times B_j \in \mathcal{P}$. We need to show that, for E,

$$\mu_0(E) = \mu_X(A)\,\mu_Y(B) \leq \sum_{j=1}^{\infty} \mu_X(A_j)\,\mu_Y(B_j) = \sum_{j=1}^{\infty} \mu_0(A_j \times B_j) = \sum_{j=1}^{\infty} \mu_0(E_j)$$

Define

$$E^x = \begin{cases} B & x \in A \\ \varnothing & x \notin A \end{cases}$$

Then,

$$E^x \subset \bigcup_{j=1}^{\infty} E_j^x$$

where

$$E_j^x = \begin{cases} B_j & x \in A_j \\ \varnothing & x \notin A_j \end{cases}$$

It follows that, for $x \in X$,

$$\mu_Y(E^x) \leq \sum_{j=1}^{\infty} \mu_Y\left(E_j^x\right).$$

This is because $x \in A$ implies

$$\mu_Y(B) \leq \sum_{j=1}^{\infty} \mu_Y\left(E_j^x\right)$$

and if $x \notin A$,

$$\mu_Y(\varnothing) = 0 \leq \sum_{j=1}^{\infty} \mu_Y\left(E_j^x\right).$$

Also, $\mu_Y(E^x) = 0$ if, for each j, $\chi_{A_j}(x) = 0$, assuming that no B_j is null. It follows that

$$\mu_X(A)\,\mu_Y(B) \leq \sum_{j=1}^{\infty} \mu_X(A_j)\,\mu_Y(B_j).$$

■

Now, by Lebesgue-Carathéodory theorem, we can extend μ_0 to a measure $\mu = \mu_X \otimes \mu_Y$, on some σ-algebra on $X \times Y$.

Example 135 *Consider $\mathfrak{m}_r$ on $\mathbb{R}^r$ and $\mathfrak{m}_s$ on $\mathbb{R}^s$. Then, $\mathfrak{m}_r \otimes \mathfrak{m}_s$ is a measure on $\mathbb{R}^{r+s}$.*

This is also the same as the canonical measure on $\mathbb{R}^{r+s}$ since they are defined on the same σ-algebra and agree on each element of the σ-algebra. A rigorous proof of this fact will be skipped.

Definition 136 *A measure μ is called **σ-finite** if X can be written as a countable union of measurable sets, each of finite measure.*

Example 137 $\mathfrak{m}_1$ *on $\mathbb{R}$, where $X_j = \left[-2^j, 2^j\right]$*

Example 138 *The counting measure is not σ-finite on $\mathbb{R}$ since any infinite set has measure infinity yet may not cover $\mathbb{R}$.*

For the rest of this section, we assume that all measures are σ-finite and complete. Let $(X, \mathfrak{A}_X, \mu_X)$ and $(Y, \mathfrak{A}_Y, \mu_Y)$ be two such measure spaces. Let $(Z, \mathfrak{A}_Z, \mu_Z)$ where $Z = X \times Y$ be the completion of the product spaces.

Proposition 139 *$(Z, \mathfrak{A}_Z, \mu_Z)$ is σ-finite.*

Proof. Let

$$X = \bigcup_{i=1}^{\infty} A_i \text{ and } Y = \bigcup_{i=1}^{\infty} B_i$$

Then, $A_i, B_i \in \mathcal{P} = \mathfrak{A}_X \times \mathfrak{A}_Y$ and since $\mu_0 = \mu|_{\mathcal{P}}$, the result follows. ■

Lemma 140 *If $(Z, \mathfrak{A}_Z, \mu_Z)$ is the complete measurable space obtained from premeasure μ_0 on $\mathcal{P} = \mathfrak{A}_X \times \mathfrak{A}_Y$, then for any $E \in \mathfrak{A}$, $\exists G$ with $E \subset G$ such that $\mu(G \backslash E) = 0$ and*

$$G = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{k,n},$$

where $A_{k,n} \in \mathcal{P}$.

Proof. That fact that we can always find a G with the given measure holds true because of completeness. If E is a bounded region, then we can have rectangles $A_{k,n}$ that determine this region. If E is unbounded, then the result is trivially true. ■

We now prove Fubini's theorem for product of characteristics functions, which is basically a variant of Cavalieri's Principle (which we state without proof):

Theorem 141 (Cavalieri's Principle) *Let $(X, \mathfrak{A}_X, \mu_X)$ and $(Y, \mathfrak{A}_Y, \mu_Y)$ be two σ-finite measure spaces. Let $(Z, \mathfrak{A}_Z, \mu_Z)$ where $Z = X \times Y$ be the completion of the product spaces. Let C be a measurable set in $\mathfrak{A}_Z$ and $C^x = \{y \in Y : (x, y) \in C\}$. Then*

1. *For almost every x, $C^x \in \mathfrak{A}_y$*
2. *The function $x \longmapsto \mu_Y(C^x)$ μ_X-is measurable*

3. The integral over C can be calculated by iterations:

$$\upsilon(C) = (\mathfrak{m}_r \otimes \mathfrak{m}_s)(C) = \int_X \mu_Y(C^x)\, d\mu_X$$

The generalization of Cavalieri's Principle to non-negative functions allows functions to be integrated in the product space, even if they are not summable.

This allows us to show that μ_0, the pre-measure obtained by the product of measure, is countably additive.

Proof. We know that μ_0 is countably subadditive by **Lemma 134**. Let $\{A_j \times B_j : j \in \mathbb{N}\}$ be a pairwise disjoint and let

$$E = A \times B = \bigcup_{n=1}^{\infty} E_j$$

where $E_j = A_j \times B_j$. Then, for each $(x, y) \in X \times Y$, note that $\chi_{A_j \times B_j}(x, y) = \chi_{A_j}(x)\,\chi_{B_j}(y)$. Thus,

$$\chi_E(x, y) = \sum_{n=1}^{\infty} \chi_{A_j}(x)\,\chi_{B_j}(y)$$

Since we have $\chi_{B_j}(y)$ μ_Y-measurable, the integral

$$\int_Y \sum_{j=1}^{\infty} \chi_{A_j}(x)\,\chi_{B_j}(y)\, d\mu_Y$$

is well-defined. Moreover, by Monotone Convergence Theorem,

$$\int_Y \lim_{n\to\infty} \sum_{j=1}^{n} \chi_{A_j}(x)\,\chi_{B_j}(y)\, d\mu_Y = \lim_{n\to\infty} \int_Y \sum_{j=1}^{n} \chi_{A_j}(x)\,\chi_{B_j}(y)\, d\mu_Y$$

so that by linearity of measure,

$$\lim_{n\to\infty} \int_Y \sum_{n=1}^{n} \chi_{A_j}(x)\,\chi_{B_j}(y)\, d\mu_Y = \lim_{n\to\infty} \sum_{n=1}^{n} \int_Y \chi_{A_j}(x)\,\chi_{B_j}(y)\, d\mu_Y$$

$$= \lim_{n\to\infty} \sum_{n=1}^{n} \chi_{A_j}(x) \int_Y \chi_{B_j}(y)\, d\mu_Y = \sum_{n=1}^{\infty} \chi_{A_j}(x) \int_Y \chi_{B_j}(y)\, d\mu_Y$$

Note that

$$\int_Y \chi_{B_j}(y)\, d\mu_Y = \mu_Y(B_j)$$

so that we have

$$\sum_{n=1}^{\infty} \chi_{A_j}(x)\, \mu_Y(B_j)$$

Now, $\chi_{A_j}(x)$ is μ_X-measurable so that

$$\int\limits_X \sum_{n=1}^{\infty} \chi_{A_j}(x)\,\mu_Y(B_j)\,d\mu_X$$

Applying the same steps as above gives us

$$\sum_{n=1}^{\infty} \mu_Y(B_j) \int\limits_X \chi_{A_j}(x)\,d\mu_X = \sum_{n=1}^{\infty} \mu_Y(B_j)\,\mu_X(A_j) = \sum_{n=1}^{\infty} \mu_X(A_j)\,\mu_Y(B_j)$$

Now,

$$\begin{aligned} & \int\limits_X \left(\int\limits_Y \chi_E(x,y)\,d\mu_Y \right) d\mu_X = \mu_Y(E^y) \int\limits_X \chi_{E^x}(x)\,d\mu_X \\ = \; & \mu_Y(E^y)\,\mu_X(E^x) = \mu_X(E^x)\,\mu_Y(E^y) = \mu_X(A)\,\mu_Y(B) = \mu_0(E) \end{aligned}$$

where

$$E^x = \begin{cases} B & x \in A \\ \varnothing & x \notin A \end{cases}$$

and

$$E^y = \begin{cases} A & x \in B \\ \varnothing & x \notin B \end{cases}$$

Thus,

$$\mu_0(E) = \sum_{n=1}^{\infty} \mu_X(A_j)\,\mu_Y(B_j) = \sum_{n=1}^{\infty} \mu_0(E_j)$$

■

Let $(X, \mathfrak{A}_X, \mu_X)$ and $(Y, \mathfrak{A}_Y, \mu_Y)$ be measurable spaces, $(Z, \mathfrak{A}_Z, \mu_Z)$ be a completion of the product space $Z = X \times Y$ and let $f : Z \longrightarrow [0, \infty]$ be a non-negative function (not necessarily μ_Z-measurable) Define f_y such that $x \xmapsto{f_y} f(x,y)$ with the integral

$$I(y) = \begin{cases} \int_X f_y d\mu_X & \text{if it exists} \\ 0 & \text{otherwise} \end{cases}$$

Then, $I(y)$ exists for almost every y. Assuming that f is μ_Z-measurable, then

$$\int_Y I(y)\,d\mu_Y = \int_Z f dv$$

Theorem 142 (Tonelli) *Let $(X, \mathfrak{A}_X, \mu_X)$ and $(Y, \mathfrak{A}_Y, \mu_Y)$ be two σ-finite measure spaces. Let $(Z, \mathfrak{A}_Z, \mu_Z)$ where $Z = X \times Y$ be the completion of the product spaces and let $f : Z \longrightarrow [0, \infty]$ be μ_Z-measurable, then*

1. *For almost every $y \in Y$, the following integral exists*

$$\int_X f_y d\mu_X$$

2. *Moreover, we have the iterated integral*

$$\int_Y \left(\int_X f_y d\mu_X \right) d\mu_Y = \int_Z f d\nu$$

We can equivalently have f_x such that $y \xmapsto{f_x} f(x, y)$, giving us

$$\int_Y \left(\int_X f_y (x) \, d\mu_X \right) d\mu_Y = \int_X \left(\int_Y f_x (y) \, d\mu_Y \right) d\mu_X.$$

Theorem 143 (Fubini) *Let $(X, \mathfrak{A}_X, \mu_X)$ and $(Y, \mathfrak{A}_Y, \mu_Y)$ be two σ-finite measure spaces. Let $(Z, \mathfrak{A}_Z, \mu_Z)$ where $Z = X \times Y$ be the completion of the product spaces. Let $f : Z \longrightarrow \mathbb{R}$ be μ_Z-summable. Then,*

1. *almost every $y \in Y$, the following integral exists*

$$\int_X f_y d\mu_X$$

2. *Moreover, we have the iterated integral*

$$\int_Y \left(\int_X f_y d\mu_X \right) d\mu_Y = \int_Z f d\nu$$

Corollary 144 *If $g(x)$ is summable on X and $h(y)$ is summable on Y, then $f(x, y) = g(x) h(y)$ is summable on Z. Moreover,*

$$\int_Z f(x, y) \, dv = \left(\int_X g d\mu_X \right) \left(\int_Y h d\mu_Y \right)$$

However, we may still not have measurability! To get that, set $g_1(x, y) = g(x)$ and $h_1(x, y) = h(y)$. In this case, g_1 and h_1 are measurable, then their product is measurable.

Problem 145 *Let $(X, \mathfrak{A}_X, \mu_X)$ and $(Y, \mathfrak{A}_Y, \mu_Y)$ be σ-finite and complete measure spaces, g be summable on X and h be summable on Y. Show that $f(x, y) = g(x) h(y)$ is summable on $X \times Y$ and that*

$$\int_Z f(x, y) \, dv = \left(\int_X g d\mu_X \right) \left(\int_Y h d\mu_Y \right)$$

Solution 146 *Note that X is $\mathfrak{A}_X$-measurable and Y is $\mathfrak{A}_Y$-measurable implies $X \times Y$ is measurable on the σ-algebra generated by $\mathfrak{A}_X \times \mathfrak{A}_Y$, say $\mathfrak{A}_Z$. Let $g_1(x,y) = g(x)$ and $h_1(x,y) = h(y)$. By Cavalieri's Principle, g_1 and h_1 are $\mathfrak{A}_Z$-measurable. Since the product of measurable functions is measurable, $f(x,y)$ is $\mathfrak{A}_Z$-measurable. We will show that $|f|$ is summable. By Toneli's Theorem,*

$$\int_Z |f(x,y)|\, dv = \int_X \int_Y |g_1(x,y)\, h_1(x,y)|\, d\mu_Y d\mu_X$$

Now, $g_1(x,y) = g(x)$ and $h_1(x,y) = h(y)$ so that

$$\begin{aligned}
&= \int_X \int_Y |g(x)\, h(y)|\, d\mu_Y d\mu_X = \int_X \int_Y |g(x)|\, |h(y)|\, d\mu_Y d\mu_X \\
&= \int_X |g(x)| \int_Y |h(y)|\, d\mu_Y d\mu_X = \int_X |g(x)| \left(\int_Y |h(y)|\, d\mu_Y \right) d\mu_X \\
&= \left(\int_X |g(x)|\, d\mu_X \right) \left(\int_Y |h(y)|\, d\mu_Y \right)
\end{aligned}$$

Now, since g and h are summable, then so is $|g|$ and $|h|$. Thus, the multiplicands in the last line are two finite numbers and, therefore, the product is finite. Thus, $|f|$ is summable so that f is summable. Thus, by Fubini's theorem,

$$\int_Z f(x,y)\, dv = \left(\int_X g d\mu_X \right) \left(\int_Y h d\mu_Y \right)$$

Problem 147 *Let $(X, \mathfrak{A}_X, \mu_X)$ and $(Y, \mathfrak{A}_Y, \mu_Y)$ be measure spaces with $X = Y = \mathbb{N}$ with μ_X and μ_Y counting measures. Define*

$$f(x,y) = \begin{cases} 2 - 2^{-x} & x = y \\ -2 + 2^{-x} & x = y + 1 \\ 0 & \textit{otherwise} \end{cases}$$

Show that f is measurable with respect to $\mu_X \otimes \mu_Y$. Does it contradict the Fubini's Theorem?

Solution 148 *We have that $\mathfrak{A}_Z = 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ since $\mathfrak{A}_X = \mathfrak{A}_Y = 2^{\mathbb{N}}$ so that every subset of $\mathbb{N} \times \mathbb{N}$ is measurable. Thus, the preimage of any measurable subset of*

$\mathbb{R}$ under f will naturally be measurable. Now,

$$
\begin{aligned}
& \int_{\mathbb{N}}\int_{\mathbb{N}} f(x,y)\, d\mu_X d\mu_Y = \int_{\mathbb{N}}\left(\int_{\mathbb{N}} f(x,y)\, d\mu_X\right) d\mu_Y \\
= & \int_{\mathbb{N}}\sum_{k=1}^{\infty} f(k,y)\, d\mu_Y = \int_{\mathbb{N}}\sum_{k=1}^{\infty} f_k(y)\, d\mu_Y \\
= & \sum_{k=1}^{\infty}\int_{\mathbb{N}} f_k(y)\, d\mu_Y = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f(k,j) \\
= & \sum_{k=1}^{\infty}\left(f(k,1)+f(k,2)+...\right) \\
= & \left\{\begin{array}{cccc} f(1,1)+ & f(2,1)+ & f(3,1)+ & ...+ \\ f(1,2)+ & f(2,2)+ & f(3,2)+ & ...+ \\ f(1,3) & f(2,3) & f(3,3)+ & ...+ \\ f(1,4) & f(2,4) & f(3,4)+ & ...+ \\ \vdots & \ddots & & \end{array}\right. \\
= & f(1,1)+f(2,1)+f(2,2)+f(3,2)+... \\
= & \left(2-2^{-1}\right)+\left(-2+2^{-2}\right)+\left(2-2^{-2}\right)+\left(-2+2^{-3}\right)+... \\
= & 3/2-7/4+7/4-15/8+15/8+...=3/2
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \int_{\mathbb{N}}\int_{\mathbb{N}} f(x,y)\, d\mu_Y d\mu_X = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f(j,k) \\
= & \sum_{k=1}^{\infty}\left(f(1,k)+f(2,k)+...\right) \\
= & \left\{\begin{array}{cccc} f(1,1)+ & f(1,2)+ & f(1,3)+ & ...+ \\ f(2,1)+ & f(2,2)+ & f(2,3)+ & ...+ \\ f(3,1) & f(2,3) & f(3,3)+ & ...+ \\ f(4,1) & f(2,4) & f(3,3)+ & ...+ \\ \vdots & \ddots & & \end{array}\right. \\
= & \sum_{k=1}^{\infty} -2^{-k}+2^{-k-1} = \sum_{k=1}^{\infty} 2^{-k-1}(-2+1) \\
= & \sum_{k=1}^{\infty} 2^{-k-1} = -\frac{1}{2}\sum_{k=1}^{\infty} 2^{-k} = -\frac{1}{2}
\end{aligned}
$$

Thus,

$$\int_{\mathbb{N}}\int_{\mathbb{N}} f(x,y)\, d\mu_X d\mu_Y \neq \int_{\mathbb{N}}\int_{\mathbb{N}} f(x,y)\, d\mu_Y d\mu_X$$

This is because f is not summable: $f^{+}(x,y) = 2-2^{-x}$ for $x = y$. Thus, $f^{+}(x,y) = g(n) = 2-2^{-n}$. On the other hand, $f^{-}(x,y) = -2+2^{-x}$ for

$x = y + 1$. That is, $f^{-}(x, y) = h(n) = -2 + 2^{-n-1}$. Note that

$$\int_{\mathbb{N}} g(n)\, d\mu = \sum_{n=1}^{\infty} 2 - 2^{-n} = \infty$$

whereas

$$\int_{\mathbb{N}} h(n)\, d\mu = \sum_{n=1}^{\infty} -2 + 2^{-n-1} = -\infty$$

Thus, Fubini's theorem is not contradicted, since the hypothesis of Fubini's theorem are not satisfied.

Problem 149 *Let $X = Y = [0, 1]$, with μ_X being the Lebesge measure and μ_Y being the counting measure, $\mathfrak{A}_X$ and $\mathfrak{A}_Y$ both the Lebesgue σ-algebras. Show that $D = \{(x, y) : x = y\} \subset X \times Y = Z$ is a measurable set with respect to $\nu = \mu_X \otimes \mu_Y$. Show that*

$$\int_Z \chi_D d\nu \neq \int_X \left(\int_Y \chi_D(x, y)\, d\mu_Y \right) d\mu_X$$

Does it contradict the Fubini Theorem and why not?

Solution 150 *Let $n \in \mathbb{N}$ and consider the square $[0, 1] \times [0, 1]$. Add $k - 1$ equidistant points (with $0 \leq k \leq n - 1$) on the diagonal. Using these, form k rectangles with heights $\frac{1}{n}, \frac{2}{n}, ..., 1$ and widths $\frac{1}{n}$. Mathematically, form the sets $\left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$. Let*

$$D_n = \bigcup_{k=0}^{n-1} \left(\left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right] \right)$$

Then,$D_n \in \mathfrak{A}_Z$ for each n. Moreover, by construction

$$D = \bigcap_{n \in \mathbb{N}} D_n$$

so that $D \in \mathfrak{A}_Z$. Next, it suffices to show that

$$\int_X \left(\int_Y \chi_D(x, y) d\mu_Y \right) d\mu_X \neq \int_Y \left(\int_X \chi_D(x, y) d\mu_X \right) d\mu_Y$$

since Fubini's theorem implies

$$\int_Y \left(\int_X \chi_D(x, y) d\mu_X \right) d\mu_Y = \int_Z \chi_D d\upsilon$$

and, therefore, the above are equal. We are essentially looking at the contrapositive of this statement. We evaluate each side of the above.

$$\begin{aligned} \int_X \left(\int_Y \chi_D(x, y) d\mu_Y \right) d\mu_X &= \int_X \mu_Y(\{y \in Y \mid y = x\})\, d\mu_X \\ = \quad \int_X \mu_Y(\{y\})\, d\mu_X &= \int_X 1 d\mu_X = 1\mu_X([0, 1]) = 1.(1 - 0) = 1 \end{aligned}$$

whereas

$$\begin{aligned}
& \int_Y \left(\int_X \chi_D(x,y) d\mu_X \right) d\mu_Y \\
= & \int_Y \mu_X \left(\{x \in X \mid x = y\} \right) d\mu_Y \\
= & \int_Y \mu_X \left(\{x\} \right) d\mu_X = \int_Y 0 d\mu_Y = 0
\end{aligned}$$

Again, this does not contradict Fubini's theorem because the space $([0,1], \mathfrak{A}_Y, \mu_Y)$ *is not* σ*-finite: assume that we can split the uncountable set* $[0,1]$ *into a countable number of countable sets. Let*

$$[0,1] = \bigcup_{n=1}^{\infty} A_n$$

where the collection $\{A_n\}_{n=1}^{\infty}$ *is pairwise disjoint and each* A_n *is countable. Since the countable union of countable sets is countable, we have that* $|[0,1]| = |\mathbb{N}|$*. That is,* $\aleph_0 = \mathfrak{c}$*, a contradiction.*

Problem 151 *Let* f, g *be two increasing functions on* $[0,1]$*, measurable with respect to* $\mathfrak{m}_1$*. Prove that*

$$\int_{[0,1]} fg d\mathfrak{m}_1 \geq \left(\int_{[0,1]} f d\mathfrak{m}_1 \right) \left(\int_{[0,1]} f d\mathfrak{m}_1 \right)$$

Solution 152 *If* f *and* g *are not summable, then* $\int_{[0,1]} f d\mathfrak{m}_1 = \int_{[0,1]} g d\mathfrak{m}_1 = \int_{[0,1]} fg d\mathfrak{m}_1 = \infty$ *and so the inequality holds. If either one of these, say* f*, is not summable, then* $\int_{[0,1]} f d\mathfrak{m}_1 = \infty \implies \int_{[0,1]} fg d\mathfrak{m}_1 = \infty$ *so that again the inequality holds. Assume that both are summable. Let* $Z = [0,1] \times [0,1]$ *and* $(x,y) \in [0,1] \times [0,1]$*. Note that*

$$\begin{aligned}
& \int_Z (f(x) - f(y))(g(x) - g(y)) d\mathfrak{m}_2 \\
= & \int_{[0,1]} \int_{[0,1]} (f(x) - f(y))(g(x) - g(y)) d\mathfrak{m}_1(x) d\mathfrak{m}_1(y) \\
= & \int_{[0,1]} \int_{[0,1]} f(x) g(x) - f(y) g(x) - f(x) g(y) + f(y) g(y) d\mathfrak{m}_1(x) d\mathfrak{m}_1(y)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[0,1]} \left[\begin{array}{c} \left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) - \left(\int_{[0,1]} f(y)\, g(x)\, d\mathfrak{m}_1(x)\right) - \\ \left(\int_{[0,1]} f(x)\, g(y)\, d\mathfrak{m}_1(x)\right) + \left(\int_{[0,1]} f(y)\, g(y)\, d\mathfrak{m}_1(x)\right) \end{array} \right] d\mathfrak{m}_1(y) \\
&= \int_{[0,1]} \left[\begin{array}{c} \left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) - \left(f(y) \int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right) - \\ g(y) \left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right) + \left(f(y)\, g(y) \int_{[0,1]} d\mathfrak{m}_1(x)\right) \end{array} \right] d\mathfrak{m}_1(y) \\
&= \int_{[0,1]} \left[\begin{array}{c} \left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) - \left(f(y) \int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right) - \\ g(y) \left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right) + (f(y)\, g(y)\, (1-0)) \end{array} \right] d\mathfrak{m}_1(y) \\
&= \int_{[0,1]} \left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) d\mathfrak{m}_1(y) - \int_{[0,1]} \left(f(y) \int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right) d\mathfrak{m}_1(y) - \\
&\quad \int_{[0,1]} g(y) \left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right) d\mathfrak{m}_1(y) + \int_{[0,1]} (f(y)\, g(y))\, d\mathfrak{m}_1(y) \\
&= \left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) \int_{[0,1]} d\mathfrak{m}_1(y) - \int_{[0,1]} \left(f(y) \int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right) d\mathfrak{m}_1(y) - \\
&\quad \int_{[0,1]} g(y) \left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right) d\mathfrak{m}_1(y) + \int_{[0,1]} (f(y)\, g(y))\, d\mathfrak{m}_1(y) \\
&= \left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) ((1-0)) - \left(\int_{[0,1]} f(y)\, d\mathfrak{m}_1(y)\right) \left(\int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right) - \\
&\quad \left(\int_{[0,1]} g(y)\, d\mathfrak{m}_1(y)\right) \left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right) + \int_{[0,1]} (f(y)\, g(y))\, d\mathfrak{m}_1(y) \\
&= \left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) - \left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right) \left(\int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right) - \\
&\quad \left(\int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right) \left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right) + \int_{[0,1]} (f(x)\, g(x))\, d\mathfrak{m}_1(x) \\
&= 2\left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) - 2\left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right) \left(\int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right)
\end{aligned}
$$

Now, if $x < y$, we have $f(x) - f(y) \leq 0$ and similarly $g(x) - g(y) \leq 0$. Thus, $(f(x) - f(y))\,(g(x) - g(y)) \geq 0$. If $x = y$, then $(f(x) - f(y))\,(g(x) - g(y)) = 0$. If $x > y$, then $f(x) - f(y) \geq 0$ and similarly $g(x) - g(y) \geq 0$. Thus, in any case, $(f(x) - f(y))\,(g(x) - g(y)) \geq 0$. Thus,

$$
(f(x) - f(y))\,(g(x) - g(y)) \geq 0 \implies \int_Z (f(x) - f(y))\,(g(x) - g(y))\, d\mathfrak{m}_2 \geq 0
$$

That is,

$$2\left(\int_{[0,1]} f(x)\, g(x)\, d\mathfrak{m}_1(x)\right) - 2\left(\int_{[0,1]} f(x)\, d\mathfrak{m}_1(x)\right)\left(\int_{[0,1]} g(x)\, d\mathfrak{m}_1(x)\right) \geq 0$$

In other words,

$$\int_{[0,1]} fg d\mathfrak{m}_1 \geq \left(\int_{[0,1]} f d\mathfrak{m}_1\right)\left(\int_{[0,1]} g d\mathfrak{m}_1\right)$$

4 Classification of Functions

4.1 Differentiability

Our ultimate goal for differentiability is to come up with nice properties of a function f such that the fundamental theorem of calculus holds. That is,

$$F(x) = \int_a^x f(t)\, dt \Longrightarrow f = F'$$

for a nice enough f. Moreover, if f is summable, then we would want F to be continuous. The natural question we should be asking is under what conditions is F differentiable? This would help answering when $F' = f$.

Problem 153 *If*

$$F(x) = \int_{(-\infty,x)} f d\mathfrak{m}_1,$$

is it true that F is continuous?

Problem 154 *Let $x_n \to x$. We need to show that $F(x_n) \to F(x)$. Note that*

$$\int_{(-\infty,x_n)} f d\mathfrak{m}_1 = \int_{\mathbb{R}} \chi_{(-\infty,x_n]} f d\mathfrak{m}_1$$

Thus, we can let $\chi_{(-\infty,x_n]} f = f_n$. Note that f_n is summable because f is, $f_n \to f$ pointwise and $|f_n| \leq f$. Thus, by Lebesgue's Dominated Convergence,

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{(-\infty,x_n]} f d\mathfrak{m}_1 = \int_{-\infty}^x f d\mathfrak{m}_1 = F(x)$$

Before we get into this business, we prove the following useful lemma.

Definition 155 *Let E be a measurable subset of $\mathbb{R}$. Let $\mathcal{F}$ be a family of closed, non-degenerate covers such that $\forall \epsilon > 0$ and $\forall x \in E$, there exists $I \in \mathcal{F}$ such that $\mathfrak{m}_1(I) < \epsilon$ and $x \in I$. F is said to be a* ***Vitali cover*** *of E.*

Theorem 156 (Vitali's Lemma) *Let E be a measurable set such that $\mathfrak{m}_1^*(E) < \infty$ and let $\mathcal{F}$ be a Vitali Cover of E. Then, we can find disjoint intervals $I_1, ..., I_n \in \mathcal{F}$ such that*

$$\mathfrak{m}_1^*\left(E \backslash \bigcup_{k=1}^{n} I_k\right) < \epsilon.$$

The proof is very geometric in nature.

Proof. We can find an open G with $E \subset G$ such that $\mathfrak{m}_1(G) < \infty$. We can always make I's from $\mathcal{F}$ smaller to put them all inside G. That is, $I \subset G$. This does not spoil our condition. This is particularly true since G is open.

If there exists a disjoint subfamily $\{I_k : 1 \leq k \leq n\} \subset \mathcal{F}$ such that

$$E \subset \bigcup_{k=1}^{n} I_k.$$

In this case, we trivially have

$$\mathfrak{m}_1^*\left(E \backslash \bigcup_{k=1}^{n} I_k\right) = 0.$$

If there is no such family, then proceed by induction: take any $I_1 \in \mathcal{F}$. We need to choose the second one. From another subfamily $\mathcal{F}_1 = \{I \in \mathcal{F} : I \cap I_1 = \varnothing\}$. This is non-empty by assumption in this case. Now, for every $I \in \mathcal{F}$ in general and in particular, in $\mathcal{F}_1$, since $I \subset G$, we must have $\mathfrak{m}_1(I) \leq \mathfrak{m}_1(G)$. Since $\mathfrak{m}_1(G) < \infty$, we must have $s_1 = \sup\{\mathfrak{m}_1(I) : I \in \mathcal{F}_1\} < \infty$. Take any $I_2 \in \mathcal{F}_1$ with $\mathfrak{m}_1(I) > s_1/2$. On the n-th step, we have $I_1, ..., I_n$ chosen. By procedure, these are pairwise disjoint. Let $\mathcal{F}_n = \{I \in \mathcal{F} : I \cap I_1 = \varnothing, ..., I \cap I_n = \varnothing\}$. This is again possible for the same reason as above with $s_n = \sup\{\mathfrak{m}_1(I) : I \in \mathcal{F}_n\} < \infty$ so that we can choose $\mathfrak{m}_1(I_{n+1}) > s_n/2$. The crucial step here relies on the fact that

$$E \backslash \bigcup_{k=1}^{n} I_k \subset \bigcup_{k=n+1}^{\infty} 5I_k$$

where $5I_k$ is the interval with the same midpoint as I_k but 5 times wider and this makes easy applications in higher dimensions. To prove this, we need to prove that 1) $\mathfrak{m}_1(I_n) \to 0$. This is because the pairwise disjoint family $\{I_n : n \in \mathbb{N}\}$ is covered by G. That is,

$$\bigcup_{k=1}^{\infty} I_k \subset G$$

so that

$$\sum_{n=1}^{\infty} \mathfrak{m}_1(I_n) \leq m_1(G) < \infty.$$

We also need to prove that

$$x \in E \backslash \bigcup_{k=1}^{n} I_k$$

so that $\exists I \in \mathcal{F}_n$ such that $x \in I$. However, $\exists I_N$ such that $I \cap I_N \neq \varnothing$. The overall strategy is to take this x and show that it is covered 5 times an interval. Assume that this is false. Then, $\forall N$, $I \cap I_N = \varnothing$, which means that $I \in \mathcal{F}_N$ for all N but this means that $s_N \geq \mathfrak{m}_1(I) \implies \mathfrak{m}_1(I_N) \geq \mathfrak{m}_1(I)/2$, a contradiction since I is fixed and $\mathfrak{m}_1(I) \not\to 0$. 3) $N \geq n$. This holds because $I \in \mathcal{F}_n$ and so it is disjoint with $I_1, ..., I_n$. Take the first N for which $I_N \cap I \neq \varnothing$. In particular, $I \cap I_1 = \varnothing, ..., I \cap I_{N-1} = \varnothing$. So, $I \in \mathcal{F}_n$ so that, the crucial thing, $\mathfrak{m}_1(I_N) > \mathfrak{m}_1(I)/2$, which justifies the 5 scale. If $x \in I$, then distance from x to the middle of I_N is $\leq \mathfrak{m}_1(I) + \mathfrak{m}_1(I_N)/2$ which is less than $5/2\mathfrak{m}_1(I_N)$ so that $x \in 5I_N$.

Now, choose n such that

$$\sum_{j=n+1}^{\infty} \mathfrak{m}_1(I_j) < \epsilon/5.$$

We can do this since the tail goes to zero. Now, the outer measure is monotonic hence

$$\mathfrak{m}_1^*\left(E \backslash \bigcup_{k=1}^{n} I_k\right) \leq \mathfrak{m}_1^*\left(\bigcup_{j=n+1}^{\infty} 5I_j\right) \leq \sum_{j=n+1}^{\infty} 5\mathfrak{m}_1^*(5I_j) < \epsilon.$$

■

The fact that Vitali's Lemma needs closed, nondegenerate intervals is crucial.

Example 157 *Let E be any finite interval and $\epsilon > 0$. For each $x \in E$, define $I_x = [x, x]$. Clearly,*

$$E \subset \bigcup_{x \in E} I_x$$

Moreover, $\mathfrak{m}_1^*(I_x) = 0$ for each x. In particular, for any $\epsilon > 0$ and for any $x \in E$, we have $0 = \mathfrak{m}_1^*(I_x) < \epsilon$ and $x \in I_x$, by construction. Then, for any n,

$$\mathfrak{m}_1^*\left(E \backslash \bigcup_{k=1}^{n} I_k\right) = \mathfrak{m}_1^*(E)$$

so that for $\epsilon \leq \mathfrak{m}_1^*(E)$, the conclusion of Vitali's lemma fails.

In fact, the Vitali lemma does not even extend to the case in which the covering collection consists of non-degenerate general (not necessarily closed) intervals.

Example 158 *Consider the set $E = (0, j)$ for some fixed $j \in \mathbb{N}$. Then, $\mathfrak{m}_1^*(E) = j < \infty$. For each $x \in E$, define intervals $B(x, r_x)$ centered at x of radius*

$$r_x = \frac{\min(j, j - x)}{2}$$

Now, consider the collection $\mathcal{F} = \{B(x, r_x) : x \in E\}$. Then, by construction,

$$E \subset \bigcup_{x \in E} B(x, r_x)$$

so that $\mathcal{F}$ is an open cover of E. Moreover, for each $x \in E$, by construction, we can find an open ball $B(x, r_x)$ such that $x \in B(x, r_x)$. Since $B(x, r_x)$ is an interval, any $x \in B(x, r_x)$ is a limit point so that for any $\epsilon > 0$; we can thus find a neighborhood I of x such that $I \subset B(x, r_x)$ and $\mathfrak{m}_1^*(I) < \epsilon$. Thus, the collection of the open balls $B(x, r_x)$ for $x \in E$ (and such associated neighborhoods I) form a Vitali cover of E. The paranthetical remark is unnecessary as $B(x, r_x) \cup I = B(x, r_x)$. Now, for any finite number $n < j$, and disjoint intervals $I_1, ..., I_n \in \mathcal{F}$ with $\mathfrak{m}_1^*(I_i) = \frac{1}{2}\min(i, i - x)$, we observe that

$$0 \leq \sum_{k=1}^{n} \mathfrak{m}_1^*(I_k) = \sum_{k=1}^{n} \min(k, k - x) \leq kn$$

By construction, $j - kn \leq j$ so that for $\epsilon < j - kn$, the conclusion of Vitali's lemma fails to hold.

Before we embark on the main topic of this section, the following problem is a useful reminder of why Lebesgue Measure's is better suited than the Borel measure.

Problem 159 *Show that any union of any collection of closed, bounded, nondegenerate intervals is measurable.*

Solution 160 *The finite and countable case is easy, since we can just appeal to the properties of the σ-algebra $\mathfrak{M}_1$. For the uncountable case, we use Vitali's lemma. Let $\mathcal{F}$ be a family of closed, bounded nondegenerate intervals $\mathfrak{I}_\alpha$, with α as the index for some uncountable indexing set J. Let $\mathcal{E} = \{\mathcal{I}_\alpha : \exists \beta \in J, \mathcal{I}_\alpha \subset \mathcal{I}_\beta\}$. Essentially, we are picking the "small" intervals from $\mathcal{F}$. Note that it is still true that*

$$F = \bigcup_{\mathfrak{I} \in \mathcal{F}} \mathfrak{I} = \bigcup_{\mathfrak{I} \in \mathcal{E}} \mathfrak{I}$$

Moreover, by construction, $\mathcal{E}$ is a Vitali Cover of F so that by Vitali's lemma, we can find disjoint intervals $I_1, ...I_k \in \mathcal{E}$ such that

$$\mathfrak{m}_1^*\left(F \backslash \bigcup_{k=1}^{n} I_k\right) < \epsilon.$$

In fact, by Regularity of Measure, we can pass to the limit to extend to the countable case to get $\{I_i : i \in \mathbb{N}\} \subset \mathcal{E}$ such that

$$\mathfrak{m}_1^*(F) = \sum_{k=1}^{\infty} \mathfrak{m}_1^*(I_k)$$

Thus, F is measurable.

Theorem 161 *If $f : (a, b) \longrightarrow \mathbb{R}$ is monotone, where the endpoints are allowed to be infinite, then f is continuous almost everywhere*

Proof. WLOG, we can assume that f is increasing, a, b are finite and we can consider the interval $[a,b]$ instead of (a,b). We can write

$$(a,b) = \bigcup_{k=1}^{\infty} \left[a + \frac{1}{k}, b - \frac{1}{k}\right]$$

For a fixed $x_0 \in (a,b)$, define

$$f\left(x_0^-\right) = \sup\{f(x) : x < x_0\} \text{ and } f\left(x_0^+\right) = \inf\{f(x) : x > x_0\}.$$

Then, $f\left(x_0^+\right) - f\left(x_0^-\right) \geq 0$ because f is increasing. f is continuous at x_0 if and only if $f\left(x_0^+\right) = f\left(x_0^-\right)$. In other words, f is not continuous at x_0 if and only if $f\left(x_0^+\right) < f\left(x_0^-\right)$. Let $J(x_0) = \left\{y : f\left(x_0^-\right) < y < f\left(x_0^+\right)\right\}$. This open set consists of points at which the function jumps, hence the choice of letter. It is obvious that $J(x_0) \subset [f(a), f(b)]$. Moreover, if $x_0 \neq x_0'$, then $J(x_0) \cap J(x_0') = \varnothing$ and so

$$[f(a), f(b)] \subset \bigcup_{x_0 \in [a,b]} J(x_0)$$

is an open cover, for which we can find finitely many subcovers. Therefore, for each n, the cardinality of the set $\left\{x_0 : \mathfrak{m}_1(J(x_0)) > \frac{1}{n}\right\}$ is finite, which implies that there are only finitely many discontinuities. ■

Let $f : [a,b] \longrightarrow \mathbb{R}$ and $x \in (a,b)$. We can define **upper derivative**

$$\overline{D}f(x) = \lim_{h\to 0^+} \sup_{|t|\leq h} \frac{f(x+t) - f(x)}{t}$$

and **lower derivative** as

$$\underline{D}f(x) = \lim_{h\to 0^+} \inf_{|t|\leq h} \frac{f(x+t) - f(x)}{t}.$$

Clearly, $\overline{D}f(x) \geq \underline{D}f(x)$. Also, $f'(x) < \infty \iff \overline{D}f(x) = \underline{D}f(x) < \infty$.

Problem 162 *Find* $\overline{D}f(0)$ *and* $\underline{D}f(0)$ *where* $f(x)$ *is defined as*

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Solution 163 *Note that*

$$\begin{aligned} \overline{D}f(0) &= \lim_{h\to 0}\left[\sup_{0<|t|\leq h} \frac{f(t) - f(0)}{t}\right] = \lim_{h\to 0}\left[\sup_{0<|t|\leq h} \frac{t\sin\frac{1}{t}}{t}\right] \\ &= \lim_{h\to 0}\left[\sup_{0<|t|\leq h} \sin\frac{1}{t}\right] = 1 \end{aligned}$$

This is because $\sin\frac{1}{t}$ *attains a supremum of* 1 *at, say,* $t = \frac{2}{\pi}h$*, which satisfies* $|t| \le h$*.*

$$\begin{aligned}\underline{D}f(0) &= \lim_{h\to 0}\left[\inf_{0<|t|\le h}\frac{f(t)-f(0)}{t}\right] = \lim_{h\to 0}\left[\inf_{0<|t|\le h}\frac{t\sin\frac{1}{t}}{t}\right] \\ &= \lim_{h\to 0}\left[\inf_{0<|t|\le h}\sin\frac{1}{t}\right] = -1\end{aligned}$$

Again, this is because $\sin\frac{1}{t}$ *attains an infimum of* -1 *at, say,* $t = \frac{3}{2\pi}h$*, which satisfies* $|t| \le h$*.*

Recall that if a continuous function f is increasing, then $Df \ge 0$. This converse holds true for the upper derivative, as well.

Proof. Let f be a continuous function on $[a,b]$ with $\overline{D}f \ge 0$ on (a,b). Let $[c,d] \subset [a,b]$ with $a \ne c \lneqq d \ne b$ and $\epsilon > 0$. Let

$$E = \left\{x \in [c,d] : \frac{f(x)-f(c)}{x-c} \le \epsilon\right\}$$

We need to show that $\sup E = d$ for every ϵ so that we can let it go to zero and get the desired result. E is well defined since

$$f(c) + \epsilon c \ge f(c) + \epsilon c$$

so that $c \in E$. Since f is continuous and $[c,d]$ is compact, $f([c,d])$ is also compact and closed. Therefore E is closed and so, $\sup E \in E$. Let $\sup E = \alpha$. If $\alpha > d$, then $\alpha \notin E$, a contradiction. If $\alpha < d$, let

$$g(h) = \sup_{0<t\le h}\frac{f(\alpha+t)-f(\alpha)}{t}$$

where $h \in (0, d-\alpha]$. Then,

$$\lim_{h\to 0^+} g(h) = \overline{D}f(\alpha)$$

By hypothesis, $\overline{D}f(\alpha) \ge 0$ so that

$$\lim_{h\to 0^+} g(h) \ge 0$$

By definition of this limit, $\forall \epsilon' > 0$, $\exists \delta$: we have that $g(h) > -\epsilon'$ whenever $h = |h| < \delta$. For $\epsilon' = \epsilon$, since $h \in (0, d-\alpha]$, $\exists \alpha_1$ with $\alpha < \alpha_1 \le d$ such that

$$\frac{f(\alpha_1)-f(\alpha)}{\alpha_1 - \alpha} > -\epsilon$$

That is,

$$f(\alpha_1) + \epsilon\alpha_1 > f(\alpha) + \epsilon\alpha \ge f(c) + \epsilon c$$

so that $\alpha_1 \in E$ and $\alpha_1 > \sup E$, a contradiction.

Thus, $\alpha = d$. In summary, for any $[c,d] \subset [a,b]$ with $c \lneqq d$ and for any $\epsilon > 0$, $f(d) \ge f(c) + \epsilon(d-c)$. Thus, we can let $\epsilon \to 0$ to get $f(d) \ge f(c)$. ■

Problem 164 *Show that if the upper and lower derivatives of f are bounded on (a, b), then there exists a constant $C > 0$ such that for every $x, y \in [a, b]$, $|f(x) - f(y)| \leq C|x - y|$*

Solution 165 *We are given the existence of two constants C_1 and C_2 such that*

$$\left|\overline{D}f(x)\right| = \left|\lim_{h\to 0}\left[\sup_{0<|t|\leq h}\frac{f(x+t)-f(x)}{t}\right]\right| = C_1 < \infty$$

and

$$\left|\underline{D}f(x)\right| = \left|\lim_{h\to 0}\left[\inf_{0<|t|\leq h}\frac{f(x+t)-f(x)}{t}\right]\right| = C_2 < \infty$$

Let $C = 2\max\{C_1, C_2\}$. Then, $\left|\overline{D}f(x)\right| \leq C/2$ and $\left|\underline{D}f(x)\right| \leq C/2$ so that

$$\left|\overline{D}f(x) + \underline{D}f(x)\right| \leq \left|\overline{D}f(x)\right| + \left|\underline{D}f(x)\right| \leq C$$

Let $x, y \in (a, b)$ with $x < y$. Then, $\exists t > 0$ such that $y - x = t$. Then,

$$\frac{f(y)-f(x)}{t} = \frac{f(x+t)-f(x)}{t}$$

so that

$$\left|\inf_{0<|t|\leq h}\frac{f(x+t)-f(x)}{t}\right| \leq \left|\frac{f(x+t)-f(x)}{t}\right| \leq \left|\sup_{0<|t|\leq h}\frac{f(x+t)-f(x)}{t}\right|$$

and so

$$\left|\underline{D}f(x)\right| \leq \left|\frac{f(y)-f(x)}{y-x}\right| \leq \left|\overline{D}f(x)\right| \leq \left|\overline{D}f(x)\right| + \left|\underline{D}f(x)\right| \leq C$$

That is,

$$\left|\frac{f(y)-f(x)}{y-x}\right| \leq C$$

Lemma 166 *Let f be an increasing function on $[a, b]$. Then,*

$$\mathfrak{m}_1^*\left(\left\{x \in (a, b) : \overline{D}f(x) > \alpha\right\}\right) \leq \frac{1}{\alpha}(f(b) - f(a))$$

and

$$\mathfrak{m}_1^*\left(\left\{x \in (a, b) : \overline{D}f(x) = \infty\right\}\right) = 0$$

Proof. If f were continuous on $[c, d] \subset [a, b]$ and differentiable on (c, d), then by the Mean Value Theorem, we have a point $x_0 \in (c, d)$ such that $f(d) - f(c) = f'(x_0)(d - c)$. If $f'(x) \geq \alpha$ on (c, d), then $f(d) - f(c) \geq \alpha(d - c)$ so the result holds if f is differentiable in (c, d).

In the general case, define $E_\alpha = \left\{x \in (a, b) : \overline{D}f(x) > \alpha\right\}$ and let $\alpha' \in (0, \alpha)$. Now, define the family $\mathcal{F} = \{I = [c, d] \subset (a, b) : f(d) - f(c) \geq \alpha'(d - c)\}$. Is this non-empty? Yes. It actually covers E_α as in the Vitali's Lemma. Indeed, take $x \in E_\alpha$. Then, $\overline{D}f(x) > \alpha$ so that $\forall \epsilon > 0$, $\exists \delta : \forall h < \delta$, $\exists t \in [-h, h]$, giving us

$$\frac{f(x+t) - f(x)}{t} > \alpha - \epsilon.$$

Take $\epsilon = \alpha - \alpha'$. Then, $\forall h < \delta$, $\exists t \in [-h, h] : f(x+t) - f(x) > t\alpha'$, giving us $I = [x, x+t]$ or $I = [x+t, x]$. We then have a disjoint collection $\{I_k = [c_k, d_k] : 1 \le k \le n\} \subset \mathcal{F}$ with

$$\mathfrak{m}_1^*\left(E_\alpha \backslash \bigcup_{j=1}^{n} I_j\right) < \epsilon$$

Note that

$$\begin{aligned}
\mathfrak{m}_1^*(E_\alpha) &\le \mathfrak{m}_1^*\left(E_\alpha \cap \bigcup_{j=1}^{n} I_j\right) + \mathfrak{m}_1^*\left(E_\alpha \backslash \bigcup_{j=1}^{n} I_j\right) \\
&\le \mathfrak{m}_1^*\left(\bigcup_{j=1}^{n} I_j\right) + \epsilon \\
&\le \sum_{j=1}^{n} (d_j - c_j) + \epsilon \\
&\le \frac{1}{\alpha'} \sum_{j=1}^{n} (f(d_j) - f(c_j)) + \epsilon \\
&\le \frac{1}{\alpha'} (f(b) - f(a)) + \epsilon
\end{aligned}$$

Since this is true for any α', we can let $\alpha' \to \alpha$ ($\epsilon \to 0$) to get what we need. Using this, we complete the proof as follows:

$$\mathfrak{m}_1^*\left(\left\{x \in (a,b) : \overline{D}f(x) = \infty\right\}\right) \le \mathfrak{m}_1^*\left(\left\{x \in (a,b) : \overline{D}f(x) > \alpha\right\}\right) \le \frac{1}{\alpha}(f(b) - f(a))$$

and let $\alpha \to \infty$. ■

And now, for a main theorem

Theorem 167 (Lebesgue's Theorem) *Let $f : (a,b) \longrightarrow \mathbb{R}$ be an increasing function. Then, $f'(x)$ exists for almost every $x \in (a,b)$*

Proof. The set $\{x : f'(x) \text{ does not exist}\} = \left\{x : \overline{D}f(x) = \infty\right\} \cup \left\{x : \overline{D}f(x) \gneq \underline{D}f(x)\right\}$

The measure of the first set is determined by **Lemma 166**. For the second one, note that

$$\left\{x : \overline{D}f(x) > \underline{D}f(x)\right\} = \bigcup_{\alpha,\beta \in \mathbb{Q}} \left\{x : \overline{D}f(x) > \alpha > \beta > \underline{D}f(x)\right\} = \bigcup_{\alpha,\beta \in \mathbb{Q}} E_{\alpha,\beta} \text{ (say)}$$

Our goal is to show that $\mathfrak{m}_1^*(E_{\alpha,\beta}) = 0$. Take an open G such that, for, $E_{\alpha,\beta} \subset G$, we have $\mathfrak{m}_1^*(G) < \mathfrak{m}_1^*(E_{\alpha,\beta}) + \epsilon$. This does not follow from the regularity of the measure! In fact, the outer measure is not even additive.

Note that
$$\mathfrak{m}_1^*(E_{\alpha,\beta}) = \inf \sum \mathfrak{m}_1((a_j, b_j)).$$

Now, let $\mathcal{F} = \{I = [c,d] \subset G : f(d) - f(c) > \beta(d-c)\}$. $\mathcal{F}$ covers $E_{\alpha,\beta}$, as in Vitali's Lemma. To show this, let $x \in E$ and $\epsilon > 0$. Then, $\overline{D}f(x) > \underline{D}f(x)$. That is,

$$\lim_{h \to 0^+} \sup_{|t| \leq h} \frac{f(x+t) - f(x)}{t} > \lim_{h \to 0^+} \inf_{|t| \leq h} \frac{f(x+t) - f(x)}{t}$$

By definition of limit and choice of ϵ, we can find $\delta > 0 : \forall h < \delta, \exists t \in [-h, h]$ such that
$$\frac{f(x+t) - f(x)}{t} > \underline{D}f(x) - \epsilon$$

Pick ϵ such that $\underline{D}f(x) - \epsilon$ is a rational number p. Then,

$$f(x+t) - f(x) > tp$$

We, therefore, have found our $[c,d] = [x, x+t]$ or $I = [x+t, x]$. Moreover, by choice of ϵ, $\mathfrak{m}_1([c,d]) = t < \epsilon$. Thus, we can take the disjoint collection $\{I_k : 1 \leq k \leq n\} \subset \mathcal{F}$ so that

$$\mathfrak{m}_1^*\left(E_{\alpha,\beta} \backslash \bigcup_{j=1}^{n} I_j\right) < \epsilon$$

where $I_k = [c_k, d_k]$, giving us

$$\begin{aligned}
\mathfrak{m}_1^*(E_{\alpha,\beta}) &\leq \sum_{j=1}^{n} \mathfrak{m}_1^*(E_{\alpha,\beta} \cap I_j) + \epsilon \\
&\leq \sum_{j=1}^{n} \mathfrak{m}_1^*\left(\left\{x \in (c_j, d_j) : \overline{D}f(x) > \alpha\right\}\right) + \epsilon \\
&\leq \frac{1}{\alpha} \sum_{j=1}^{n} (f(d_j) - f(c_j)) + \epsilon \text{ (by } \textbf{Lemma 166}\text{)} \\
&\Longrightarrow \sum_{j=1}^{n} (f(d_j) - f(c_j)) \geq \alpha \mathfrak{m}_1^*(E_{\alpha,\beta}) - \alpha\epsilon
\end{aligned}$$

On the other hand,

$$\sum_{j=1}^{n} (f(d_j) - f(c_j)) < \beta \sum_{j=1}^{n} (d_j - c_j) = \beta \sum_{j=1}^{n} \mathfrak{m}_1(I_j) < \beta \mathfrak{m}_1(G) < \beta \mathfrak{m}_1^*(E_{\alpha,\beta}) + \beta\epsilon$$

giving us $\alpha\mathfrak{m}_1^*(E_{\alpha,\beta}) - \alpha\epsilon \leq \beta\mathfrak{m}_1^*(E_{\alpha,\beta}) + \beta\epsilon$. That is,

$$\mathfrak{m}_1^*(E_{\alpha,\beta}) \leq \epsilon\frac{\alpha+\beta}{\alpha-\beta}.$$

Letting $\epsilon \to 0$, we get the desired result. ■

Corollary 168 *Let $f : [0,b] \longrightarrow \mathbb{R}$ be an increasing function. Then,*

$$\int\limits_{[a,b]} f' d\mathfrak{m}_1 \leq f(b) - f(a)$$

Proof. Set $f(x) = f(b)$ for $x \geq b$. Then, f' exists on $E \subset [a,b+1]$ and $\mathfrak{m}_1([a,b+1] \backslash E) = 0$ so that $f'(x) = \overline{D}f(x) \geq 0$ for $x \in E$. For this proof and onwards, we introduce the following notation: let

$$D_h f(x) = \frac{f(x+h) - f(x)}{h}$$

For a fixed $h \leq 1$, $D_h f$ is measurable on $[a,b]$. If $x \in E$, we have $D_{1/n}f(x) \to f'(x)$ (we can only take the limit when we have a countable set, as in a sequence). Thus, f' is measurable on E. Since $\mathfrak{m}_1$ is complete, by **Problem 46**, f' is measurable on $[a,b]$. Therefore,

$$\int\limits_{[a,b]} f' d\mathfrak{m}_1$$

is legal. Now, by Fatou's Lemma,

$$\varliminf_{n\to\infty} \int\limits_{[a,b]} D_{1/n} f d\mathfrak{m}_1 \geq \int\limits_{[a,b]} \varliminf_{n\to\infty} D_{1/n} f d\mathfrak{m}_1 = \int\limits_{[a,b]} f' d\mathfrak{m}_1.$$

For $h = 1/n$, we therefore have

$$\int\limits_{[a,b]} D_h f d\mathfrak{m}_1 = \int\limits_{[a,b]} \frac{f(x+h) - f(x)}{h} d\mathfrak{m}_1.$$

We can now use linearity of the measure, which applies only when f is summable, to give

$$\int\limits_{[a,b]} \frac{f(x+h) - f(x)}{h} d\mathfrak{m}_1 = 1/h\left(\int\limits_{[a,b]} f(x+h)\, d\mathfrak{m}_1 - \int\limits_{[a,b]} f(x)\, d\mathfrak{m}_1\right). \quad (8)$$

First note that

$$\int\limits_{[a,b]} g(x+h)\, d\mathfrak{m}_1(x) = \int\limits_{[a+h,b+h]} g(x)\, d\mathfrak{m}_1(x)$$

for any measurable $g : X \longrightarrow \mathbb{R}$ and $[a,b] \subset X$. To see this, let $t = x + h$. Then, $x = a \implies t = a + h$ and, similarly, $x = b \implies t = b + h$. Replacing the dummy variable t with x gives us the required result and, therefore, modifies our integral in **Eq** (8) to

$$\begin{aligned}
& 1/h \left(\int\limits_{[a+h,b+h]} f(x)\, d\mathfrak{m}_1 - \int\limits_{[a,b]} f(x)\, d\mathfrak{m}_1 \right) \\
= \;& 1/h \left(\int\limits_{[a+h,b]} f(x)\, d\mathfrak{m}_1 + \int\limits_{[b,b+h]} f(x)\, d\mathfrak{m}_1 - \int\limits_{[a,b]} f(x)\, d\mathfrak{m}_1 \right) \\
= \;& 1/h \left(\int\limits_{[a+h,b]} f(x)\, d\mathfrak{m}_1 + \int\limits_{[b,b+h]} f(x)\, d\mathfrak{m}_1 + \int\limits_{[a,a+h]} f(x)\, d\mathfrak{m}_1 - \int\limits_{[a,a+h]} f(x)\, d\mathfrak{m}_1 - \int\limits_{[a,b]} f(x)\, d\mathfrak{m}_1 \right) \\
= \;& 1/h \left(\left(\int\limits_{[a,a+h]} f(x)\, d\mathfrak{m}_1 + \int\limits_{[a+h,b]} f(x)\, d\mathfrak{m}_1 \right) - \int\limits_{[a,b]} f(x)\, d\mathfrak{m}_1 + \int\limits_{[b,b+h]} f(x)\, d\mathfrak{m}_1 - \int\limits_{[a,a+h]} f(x)\, d\mathfrak{m}_1 \right) \\
= \;& 1/h \left(\int\limits_{[b,b+h]} f(x)\, d\mathfrak{m}_1 - \int\limits_{[a,a+h]} f(x)\, d\mathfrak{m}_1 \right)
\end{aligned}$$

because $[a, b+h] = [a, a+h] \cup [a+h, b] \cup [b, b+h]$. Now, we can use $f(x) = f(b)$ to give

$$1/h \left(hf(b) - \int\limits_{[a,a+h]} f d\mathfrak{m}_1 \right) \leq \frac{1}{h} \left(hf(b) - hf(a) \right) = f(b) - f(a).$$

Thus,

$$\int\limits_{[a,b]} D_{1/n} f d\mathfrak{m}_1 \leq f(b) - f(a)$$

so that

$$\varliminf_{n\to\infty} \int\limits_{[a,b]} D_{1/n} f d\mathfrak{m}_1 \leq f(b) - f(a).$$

■

This inequality is very sharp!

Example 169 *Consider*

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is continuous and finite but not monotone. Moreover,

$$\int_{[0,1]} f' d\mathfrak{m}_1 = \infty.$$

Thus, f' is not summable so that monotonicity is crucial, even when the function above is differentiable at $x \neq 0$.

4.2 Functions of Bounded Variation

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function and $P = \{x_j : 0 \leq j \leq n\}$ be a partition of the interval $[a,b]$. Define

$$V(f,P) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|$$

Without the absolute value, this is just a telescopic sum. With it, the bounded variation gives a sense of how oscillating the function is. Some partitions do that and some don't. To find out how bad these oscillations can get, we take the supremum over partitions. This is called the **total variation** $TV(f)$:

$$TV(f) = \sup_P V(f,P),$$

which may be denoted by $TV\left(f_{[a,b]}\right)$ since the total variation is defined on a particular domain for a function f. If $TV(f) < \infty$, then f is of bounded variation.

Problem 170 $TV(f) = TV(-f)$

Solution 171 *Let P be a partition of the largest closed subset of domain of f. Then,*

$$V(-f,P) = \sum_{j=1}^{n} |(-f)(x_j) - (-f)(x_{j-1})| = \sum_{j=1}^{n} |-f(x_j) + f(x_{j-1})| = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| = V(-f,P)$$

Since supremum is taken over all partitions, the result follows.

Example 172 *Let f be a monotonic increasing function. Then, $TV\left(f_{[a,b]}\right) = f(b) - f(a)$. This is because $f(x_j) - f(x_{j-1}) > 0$ for $x_{j-1} < x_j$ so that $|f(x_j) - f(x_{j-1})| = f(x_j) - f(x_{j-1})$ and the sum becomes telescoping.*

Example 173 *Another good family of functions called Lipschitz are defined by a constant C such that, for any $x, y \in [a,b] : |f(x) - f(y)| \leq C|x-y|$. The absolute value function is one example. Lipschitz functions are all continuous and differentiable almost everywhere. Note that*

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq C \sum_{j=1}^{n} |x_j - x_{j-1}| = C(b-a)$$

hence Lipshcitz functions are of bounded variation over a bounded domain.

Example 174 *We have seen in **Problem 164** that if functions with bounded upper and lower derivatives are Lipschitz. Thus, functions with bounded upper and lower derivatives are also of bounded variation.*

Example 175 *Consider the function*

$$f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

with partition $P_N = \left\{0, \frac{1}{2N}, \frac{1}{3N}, ..., \frac{1}{3}, \frac{1}{2}, 1\right\}$*. Computing the variation tells us that each consecutive point applied on* f *give us zero and 1 on each subsequent point for the cosine part. Thus,* $V(f, P_N) = \frac{1}{N} + ... + 1$*. Therefore, the function* f *is not of bounded variation.*

Proposition 176 $TV\left(f_{[a,b]}\right) = TV\left(f_{[a,c]}\right) + TV\left(f_{[c,b]}\right)$ *if* $a \leq c \leq b$

Proof. Let $P = \{x_j : 0 \leq j \leq n\}$ be a partition of the interval $[a, b]$. Then, if $\exists k$ such that $x_k = c$, then by triangular inequality,

$$\begin{aligned} V\left(f_{[a,b]}, P\right) &= \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq \sum_{j=1}^{k} |f(x_j) - f(x_{j-1})| + \sum_{j=k}^{n} |f(x_j) - f(x_{j-1})| \\ &= V\left(f_{[a,c]}, P'\right) + V\left(f_{[c,b]}, P''\right) \end{aligned}$$

for some partition P' of $[a, c]$ and P'' of $[c, b]$. Since this holds for any P, we have $TV\left(f_{[a,b]}\right) \leq TV\left(f_{[a,c]}\right) + TV\left(f_{[c,b]}\right)$. Conversely, note that any paritition P of $[a, c]$ and P' of $[c, b]$ gives rise to a partition $P \cup P'$ of $[a, b]$. Thus, $V\left(f_{[a,c]}, P\right) + V\left(f_{[c,b]}, P\right) \leq V\left(f_{[a,b]}, P \cup P'\right)$. It follows that $TV\left(f_{[a,b]}\right) \geq TV\left(f_{[a,c]}\right) + TV\left(f_{[c,b]}\right)$. ∎

Theorem 177 (Jordan Decomposition) $TV(f) < \infty \iff f(x) = f_1(x) - f_2(x)$ *where* f_1, f_2 *are both increasing*

Proof. ($\implies$) Let $\varphi(x) = TV\left(f_{[a,x]}\right)$. If $x > y$, then, $TV\left(f_{[a,x]}\right) - TV\left(f_{[a,y]}\right) = TV\left(f_{[y,x]}\right) \geq 0$. That is, $\varphi(x) \geq \varphi(y)$. That is, φ is an increasing function. Now let $\psi(x) = f(x) + \varphi(x)$. $\psi(x)$ is also monotone, regardless of the behaviour of f: if $x > y$, then consider the partition $P = \{x, y\}$. Then, $f(x) - f(y) \leq |f(x) - f(y)| \leq TV\left(f_{[y,x]}\right) = TV\left(f_{[a,x]}\right) - TV\left(f_{[a,y]}\right)$. Similarly for $f(y) - f(x) \leq |f(x) - f(y)| \leq TV\left(f_{[y,x]}\right) = TV\left(f_{[a,x]}\right) - TV\left(f_{[a,y]}\right)$. In summary, $f(y) + TV\left(f_{[a,y]}\right) \leq f(x) + TV\left(f_{[a,x]}\right)$ so that $\psi(y) \leq \psi(x)$. That is, we have found ourselves an expression $f(x) = \psi(x) - \varphi(x)$, where ψ and φ are both increasing.

($\impliedby$) For any partition $P = \{x_j : 0 \leq j \leq n\}$ of the interval $[a, b]$,

$$\begin{aligned} \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| &= \sum_{j=1}^{n} |f_1(x_j) - f_1(x_{j-1}) + f_2(x_{j-1}) - f_2(x_j)| \\ &\leq \sum_{j=1}^{n} |f_1(x_j) - f_1(x_{j-1})| + \sum_{j=1}^{n} |f_2(x_j) - f_2(x_{j-1})| \\ &= f_1(b) - f_1(a) + f_2(b) - f_2(a) < \infty \end{aligned}$$

■

Corollary 178 *$TV(f) < \infty \Longrightarrow f'$ exists almost everywhere*

Proof. By Jordan's Decomposition, $f = f_1 - f_2$ where f_1 and f_2 are both monotone. By Lebesgue's Theorem, both are differentiable almost everywhere, as is their difference. ■

Corollary 179 *$TV(f) < \infty \Longrightarrow f'$ is summable*

Proof. Follows directly by Jordan Decomposition and **Corollary 168**. ■

Problem 180 *Let $TV(f) < \infty$ on $[a,b]$ and define $\varphi(x) = TV\left(f_{[a,x]}\right)$. Show that $|f'| \leq \varphi'$ almost everywhere on $[a,b]$. Deduce that*

$$\int_{[a,b]} |f'| \, d\mathfrak{m}_1 \leq TV(f)$$

Solution 181 *Since f is bounded variation, then f' exists almost everywhere. Moreover, since φ is increasing, φ' exists almost everywhere on (a,b). Let y be such a point where $f'(y)$ and $\varphi'(y)$ are defined. Let $x \in [a,b]$ with $x > y$. Consider the partition $P = \{x, y\}$. Then, $f(x) - f(y) \leq |f(x) - f(y)| \leq TV\left(f_{[y,x]}\right) = TV\left(f_{[a,x]}\right) - TV\left(f_{[a,y]}\right)$. Since $x > y$, there exists $h > 0$ such that $x = h + y$. Then,*

$$\frac{|f(h+y) - f(y)|}{h} \leq \frac{TV\left(f_{[a,y+h]}\right) - TV\left(f_{[a,y]}\right)}{h} = \frac{\varphi(y+h) - \varphi(y)}{h}$$

Since this holds for any such x, we can let $h \to 0$ to get $|f'(y)| \leq \varphi'(y)$. Since y was arbitrary, therefore $|f'| \leq \varphi'$ almost everywhere.

*Now, since φ is continuous and increasing on $[a,b]$, as a real-valued function. Then by **Corollary 168**,*

$$\int_{[a,b]} \varphi' d\mathfrak{m}_1 \leq \varphi(b) - \varphi(a)$$

From $|f'| \leq \varphi'$, we have

$$\begin{aligned} \int_{[a,b]} |f'| \, d\mathfrak{m}_1 &\leq \int_{[a,b]} \varphi' d\mathfrak{m}_1 \leq \varphi(b) - \varphi(a) \\ &= TV\left(f_{[a,b]}\right) - TV\left(f_{[a,a]}\right) = TV\left(f_{[a,b]}\right) = TV(f) \end{aligned}$$

Definition 182 *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function. Then, f is said to be* ***absolutely continuous*** *if $\forall \epsilon > 0$, $\exists \delta > 0$ such that, if $\{(a_i, b_i) : 1 \leq i \leq n\}$ is a family of disjoint intervals, then*

$$\sum_{j=1}^{n} b_j - a_j < \delta \Longrightarrow \sum_{j=1}^{n} |f(b_j) - f(a_j)| < \epsilon$$

These intervals may not be a partition of $[a,b]$. Note that continuity on $[a,b]$ may imply uniform continuity but uniform continuity does not imply absolute continuity. Also note that the sum of two absolutely continuous functions is continuous. Skipping this, we move to

Proposition 183 *Let f,g be absolutely continuous on $[a,b]$. Show that the function $f.g$ defined by $(f.g)(x) = f(x)\,g(x)$ is absolutely continuous.*

Proof. First, we prove that $f.g$ is continuous. Let $x_0 \in [a,b]$. Since f,g are both continuous, for any $\epsilon > 0$, we have δ_1, δ_2 and δ_3 such that

$$|x - x_0| < \delta_1 \implies |f(x) - f(x_0)| < \frac{\epsilon}{2\left(|g(x_0)| + \epsilon\right)}$$

$$|x - x_0| < \delta_2 \implies |g(x) - g(x_0)| < \frac{\epsilon}{2\,|f(x_0)|}$$

and

$$|x - x_0| < \delta_3 \implies |g(x) - g(x_0)| < |g(x_0)| + \epsilon$$

Take $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then,

$$\begin{aligned}
& |(f.g)(x) - (f.g)(x_0)| \\
= \;& |f(x)\,g(x) - f(x_0)\,g(x_0)| = |f(x)\,g(x) - f(x_0)\,g(x) + f(x_0)\,g(x) - f(x_0)\,g(x_0)| \\
\leq \;& |f(x)\,g(x) - f(x_0)\,g(x)| + |f(x_0)\,g(x) - f(x_0)\,g(x_0)| \\
= \;& |f(x) - f(x_0)|\,|g(x)| + |f(x_0)|\,|g(x) - g(x_0)| < \frac{\epsilon}{2\left(|g(x_0)| + \epsilon\right)}\,|g(x)| + |f(x_0)|\,\frac{\epsilon}{2\,|f(x_0)|} \\
< \;& \frac{\epsilon}{2\left(|g(x_0)| + \epsilon\right)}\left(|g(x_0)| + \epsilon\right) + |f(x_0)|\,\frac{\epsilon}{2\left(|f(x_0)|\right)} = \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

■

Since the function $f.g$ is continuous on a compact domain $[a,b]$, it achives its maximum and minimum by Extreme Value Theorem. Thus, $f.g$ is bounded. Let $|f.g| \leq M$

Now, given $\epsilon > 0$, we know that $\exists \delta_1, \delta_2$ (distinct from above proof!) such that $\{(a_i, b_i) : 1 \leq i \leq n\}$ is a family of disjoint intervals with

$$\sum_{j=1}^{n} b_j - a_j < \delta_1 \Longrightarrow \sum_{j=1}^{n} |f(b_j) - f(a_j)| < \epsilon/2nM$$

and

$$\sum_{j=1}^{n} b_j - a_j < \delta_2 \Longrightarrow \sum_{j=1}^{n} |g(b_j) - g(a_j)| < \epsilon/2nM$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for each j,

$$\begin{aligned}
& |f(b_j)\,g(b_j) - f(a_j)\,g(a_j)| \\
= \;& |f(b_j)\,g(b_j) - f(b_j)\,g(a_j) + f(b_j)\,g(a_j) - f(a_j)\,g(a_j)| \\
\leq \;& |f(b_j)\,g(b_j) - f(b_j)\,g(a_j)| + |f(b_j)\,g(a_j) - f(a_j)\,g(a_j)| \\
= \;& |f(b_j)|\,|g(b_j) - g(a_j)| + |f(b_j) - f(a_j)|\,|g(a_j)| \\
< \;& \frac{|f(b_j)|\,\epsilon}{2nM} + \frac{|g(a_j)|\,\epsilon}{2nM} \leq \frac{M\epsilon}{2nM} + \frac{M\epsilon}{2nM} = \epsilon/n
\end{aligned}$$

so that

$$\sum_{i=1}^{n} |f(b_j) g(b_j) - f(a_j) g(a_j)| < \frac{n\epsilon}{n} = \epsilon$$

Thus, $f.g$ is absolutely continuous.

Example 184 *A good example is a Lipschitz function. Let* $|f(x) - f(y)| \leq C|x-y|$. *Then,*

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| \leq C \sum_{j=1}^{n} |b_j - a_j|$$

so that we can take $\delta = \epsilon / C$.

Example 185 *An absolutely continuous function is Lipschitz if and only if* $|f'|$ *is bounded. To show this, let* $|f'| \leq c$ *and* $x, y \in \mathcal{D}(f)$, *domain of* f. *Consider the interval* $[x, y]$. *Then,* f *is absolutely continuous on* $[x, y]$ *so that*

$$\left| \int_x^y f' d\mathfrak{m}_1 \right| = |f(y) - f(x)| \leq \int_x^y |f'| \, d\mathfrak{m}_1 \leq c\mathfrak{m}_1 [x, y] = c|y - x|$$

Converse follows from ***Example 174***.

Theorem 186 *Let* f *be absolutely continuous. Then,* f *is of bounded variation. Moreover,* $f = f_1 - f_2$ *where* f_1, f_2 *are both increasing and absolutely continuous.*

Proof. Take δ for $\epsilon = 1$. Split $[a, b]$ into N intervals of length $< \delta$. $N \approx \frac{b-a}{\delta}$. These intervals are $[a_k, b_k]$. Then,

$$TV(f) = \sum_{k=1}^{N} TV\left(f_{[a_k, b_k]}\right)$$

If $P = \{x_k : 0 \leq k \leq n\}$ is a partition of $[a_j, b_j]$, then $\{(x_k, x_{k+1}) : 0 \leq k \leq n-1\}$ is a disjoint collection and

$$\sum_{k=1}^{n} |x_{k-1} - x_k| = |b_j - a_j| < \delta \Longrightarrow V(f, P) < 1 \Longrightarrow TV\left(f_{[a_j, b_j]}\right) \leq 1 \Longrightarrow TV(f) \leq N$$

Now, denote $f_2(x) = \varphi(x) = TV\left(f_{[a,x]}\right)$ and $f_1(x) = f(x) + TV\left(f_{[a,x]}\right) = \psi(x)$. It suffices to show that $\varphi(x)$ is absolutely continuous. Take $\delta = \epsilon/2$. Assume that

$$\sum_{k=1}^{n} b_k - a_k < \delta$$

where $\{(a_k, b_k) : 1 \leq k \leq n\}$ is a disjoint collection of intervals. Now, consider the partition P_k of (a_k, b_k). Then,

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} V(f, P_k) < \epsilon/2$$

so that
$$\sum_{k=1}^{n} TV\left(f_{[a_k,b_k]}\right) \leq \epsilon/2 < \epsilon$$
Now, by the linearity $TV\left(f_{[a_k,b_k]}\right) = TV\left(f_{[a,b_k]}\right) - \left(f_{[a,a_k]}\right) = \varphi\left(b_k\right) - \varphi\left(a_k\right)$ so that
$$\sum_{k=1}^{n} \left|\varphi\left(b_k\right) - \varphi\left(a_k\right)\right| < \epsilon$$
■

Thus, in addition to being absolutely continuous, we can always assume that our function is increasing. However, note that there exists a function of bounded variation which is not absolutely increasing. For example, any monotone function.

Problem 187 *Let f be continuous on $[0,1]$ and absolutely continuous on $[\epsilon,1]$ for each $0<\epsilon<1$. Show that f may not be absolutely continuous on $[0,1]$, but it is if f in increasing.*

Solution 188 *For the first part, we want to have a function continuous on $[0,1]$, absolutely continuous on $[\epsilon,1]$ but not absolutely continuous on $[0,1]$. Consider the function*
$$f(x) = \begin{cases} x\sin\frac{1}{x} & x\in(0,1] \\ 0 & x=0 \end{cases}$$
f is continuous on $(0,1]$ clearly, as it is the product of two continuous functions on $(0,1]$. f is also continuous at 0: note that
$$|f(x)-f(0)| = \left|x\sin\frac{1}{x}\right| \leq |x|$$
since $\left|\sin\frac{1}{x}\right| \leq 1$. Then,
$$\lim_{x\to 0}|f(x)-f(0)| \leq 0 \implies \lim_{x\to 0} f(x) = f(0)$$
Moreover, f is differentiable on $[\epsilon,1]$ for any $\epsilon>0$ and, by the product rule,
$$\begin{aligned} f'(x) &= \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right) \implies |f'(x)| = \left|\sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)\right| \\ &\leq \left|\sin\left(\frac{1}{x}\right)\right| + \left|\frac{1}{x}\cos\left(\frac{1}{x}\right)\right| \leq 1+\frac{1}{x} \leq 1+\frac{1}{\epsilon} \end{aligned}$$
*By **Example 185**, f is Lipschitz on $[\epsilon,1]$. Therefore, by **Example 184**, f is absolutely continuous on $[\epsilon,1]$. It remains to show that f is not absolutely continuous on $[0,1]$. We can accomplish this by showing that f is not of bounded variation. Let $\epsilon>0$. Then, $\exists n$ such that $\epsilon>1/n$. Consider the partition P*

defined by $x_{n+1} = 0 \leq \frac{1}{n\pi+\pi/2} \leq \frac{1}{(n-1)\pi+\pi/2} \leq .... \leq \frac{1}{\pi+\pi/2} \leq 1 = x_0$. *Let* $x_j = 1/(j\pi+\pi/2)$ *for* $j \in \{1,...,n\} = J$. *Then,*

$$\sin x_j = \begin{cases} 1 & j \in J \text{ is even} \\ -1 & j \in J \text{ is odd} \\ \sin 1 & j = 0 \\ 0 & j = n+1 \end{cases}$$

so that

$$f(x_j) = \begin{cases} x_j & j \text{ is even and } j \neq 0 \\ -x_j & j \text{ is odd and } j \neq n+1 \\ \sin 1 & j = 0 \\ 0 & j = n+1 \end{cases}$$

Then,

$$\begin{aligned} \sum_{j=1}^{n+1} |f(x_j) - f(x_{j-1})| &= |x_1 - 0| + |x_2 - x_3| + ... + |\sin 1 - x_n| \\ &= x_1 + x_2 + x_3 + ... + x_n + \sin 1 \\ &= \sin 1 + \sum_{j=1}^{n} 1/(j\pi+\pi/2) \end{aligned}$$

The sum on the right is the (displaced) harmonic series, which diverges as $n \to \infty$. *Hence the function is not of bounded variation.*

Problem 189 *Let* f *be continuous and increasing on* $[0,1]$ *and absolutely continuous on* $[\epsilon, 1]$ *for each* $0 < \epsilon < 1$. *Show that* f *is absolutely continuous.*

Solution 190 *Since* f *is increasing, given* $\epsilon > 0$, *we must have* $f(\epsilon) - f(0) > 0$. *Since* f *is continuous on* $[0,1]$, $\forall \xi > 0$ $\exists \delta'$ *such that* $|\epsilon - 0| = \epsilon < \delta' \Longrightarrow |f(\epsilon) - f(0)| = f(\epsilon) - f(0) < \xi/2$. *Let* $n \in \mathbb{N}$ *and* $\mathcal{I} = \{1,2,...,n\}$. *Since* $\xi/2 > 0$ *and* f *is absolutely continuous on* $[\epsilon, 1]$, $\exists \delta'' > 0$ *such that, for a family of disjoint intervals* $\{(a_i, b_i) : i \in \mathcal{I}\}$, *each a subset of* $[\epsilon, 1]$,

$$\sum_{j=1}^{n} |b_j - a_j| < \delta'' \Longrightarrow \sum_{j=1}^{n} |f(b_j) - f(a_j)| < \xi/2$$

Consider a finite family $\{(a_i, b_i) : 1 \leq i \leq m\}$ *of disjoint subintervals of* $[0,1]$. *Let* $I = \{i : (a_i, b_i) \subset [\epsilon, 1]\}$ *and* $J = \{i : (a_i, b_i) \subset [0, \epsilon]\}$. *Note that if* $\epsilon \in (a_i, b_i)$, *then* $a_i < \epsilon < b_i$ *so that* $f(a_i) < f(\epsilon) < f(b_i)$. *Moreover,* $|f(b_i) - f(\epsilon)| = f(b_i) - f(\epsilon)$ *and* $|f(\epsilon) - f(a_i)| = f(\epsilon) - f(a_i)$ *so that* $|f(b_i) - f(\epsilon)| + |f(\epsilon) - f(a_i)| = |f(b_i) - f(a_i)|$. *Thus, the presence of* ϵ *in an interval makes no difference in the sum we are looking for, so we can assume that* $I \cup J = \{1, 2, ..., m\}$. *Now, because* f *is increasing, we must have*

$$\sum_{j \in J} |b_j - a_j| < \frac{\delta'}{2} \quad \Longrightarrow \sum_{j \in J} |f(b_j) - f(a_j)| \leq f(\epsilon) - f(0) < \xi/2$$

Moreover, WLOG, we can assume that $|I| = |\mathcal{I}|$. Thus, we can re-arrange terms in J to get $I = \{1, 2, ..., n'\}$. Now let $\frac{\delta}{2} = \max\left\{\delta', \delta''\right\}$. Then, since $\epsilon \geq a_j, b_j$ for each $j \in J$, we must have

$$\sum_{j \in J} |b_j - a_j| < \frac{\delta}{2} \text{ and } \sum_{i \in I} |b_i - a_i| < \frac{\delta}{2}$$

so that

$$\sum_{i=1}^{m} |b_j - a_j| < \delta \implies \sum_{i=1}^{m} |f(b_j) - f(a_j)| < \frac{\xi}{2} + \frac{\xi}{2} = \xi$$

That is, for any $\xi > 0$ we can find a δ such that for any finite family $\{(a_i, b_i) : 1 \leq i \leq m\}$ of disjoint subintervals of $[0, 1]$, the above holds.

If f is absolutely continuous on $[a, b]$, we can always extend the domain to $[a, b + c]$ for some constant c and define $f(x) = f(b)$ for $x > b$.

Definition 191 *Let $\mathcal{F}$ be a family of functions. $\mathcal{F}$ is said to be* ***uniformly summable*** *if $\forall \epsilon > 0$, $\exists \delta > 0$ such that*

$$\mathfrak{m}_1(A) < \delta \Longrightarrow \int_A |g| \, d\mathfrak{m}_1 < \epsilon$$

for all $g \in \mathcal{F}$

Theorem 192 *If f is absolutely continuous on $[a, b]$, then $\{D_h f : 0 \leq h \leq 1\}$ is uniformly summable. That is, $\forall \epsilon > 0$, $\exists \delta > 0$ such that*

$$\mathfrak{m}_1(A) < \delta \implies \int_A |D_h f| \, d\mathfrak{m}_1 < \epsilon \forall h$$

Proof. Observe that we can assume that f is increasing so that $|D_h f| = D_h f$. Let $\epsilon > 0$. We need to find a $\delta > 0$ such that

$$\int_A D_h f d\mathfrak{m}_1 < \epsilon$$

whenever $\mathfrak{m}_1(A) < \delta$. Observe further that we have a G_δ set G with $A \subset G$ and $\mathfrak{m}_1(G \backslash A) = 0$. Therefore, WLOG, we can assume that A is a G_δ set. Thus, A is the countable intersection of open sets $\{G_i : i \in \mathbb{N}\}$, with $G_{n+1} \subset G_n$ and $\mathfrak{m}_1(G_1) < \infty$ and G_n is a finite disjoint union of open intervals. This follows from the construction in the proof of regularity of Lebesgue Measure because $\mathfrak{m}_1(G_n) \to \mathfrak{m}_1(G)$ and

$$\int_{G_n} D_h f \to \int_G D_h f$$

as $n \to \infty$. Now,

$$A = \bigcup_{i=1}^{n} (a_i, b_i)$$

where intervals are pairwise disjoint and

$$\sum_{i=1}^{n}(b_i - a_i) < \delta$$

To calculate

$$\int_{a_j}^{b_j} D_h f d\mathfrak{m}_1,$$

we observe that

$$g_j(t) = f(b_j + t) - f(a_j + t) = \frac{1}{h}\int_0^h g_j d\mathfrak{m}_1$$

so that

$$\int_A D_h f d\mathfrak{m}_1 = \frac{1}{h}\int_0^h \left(\sum_{i=1}^{n} g_i\right) d\mathfrak{m}_1$$

and for every t,

$$\sum_{i=1}^{n} g_i = \sum_{i=1}^{n} f(b_j + t) - f(a_j + t) < \epsilon,$$

by absolute continuity of f, where ϵ does not depend on t. ■

Lemma 193 *Let $\{f_i : i \in \mathbb{N}\}$ be a family of real-valued, uniformly summable functions. If $f_n \to f$ almost everywhere, then for every A with $\mathfrak{m}_1(A) < \infty$, f is summable on A and*

$$\int_A f_n d\mathfrak{m}_1 \to \int_A f d\mathfrak{m}_1$$

Proof. Since the measure of A is finite, we can split A into a union of N A_k's, where $\mathfrak{m}_1(A_k) < \delta$ and δ comes from the uniform summability of $\epsilon = 1$. Then, by Fatou's Lemma,

$$\int_A |f| \, d\mathfrak{m}_1 \le \underline{\lim} \int_A |f_n| \, d\mathfrak{m}_1 \le \underline{\lim} \sum_{k=1}^{N} \int_{A_k} |f_n| \, d\mathfrak{m}_1.$$

Note that

$$\int_{A_k} |f_n| \, d\mathfrak{m}_1 \le 1 \implies \int_A |f| \, d\mathfrak{m}_1 \le N$$

so that f is summable on A, $f \in \mathcal{L}$

Now, WLOG, assume that $f = 0$ and $f_n \ge 0$ (otherwise take $|f_n - f|$). By Egoroff's Theorem, $\exists$ a set $A_0 \subset A$ such that $\mathfrak{m}_1(A_0) < \delta$ and f_n uniformly converges to 0, $f_n \rightrightarrows 0$ on $A \backslash A_0$. Thus, for $n \ge N_0$,

$$\int_A f_n d\mathfrak{m}_1 = \int_{A \backslash A_0} f_n d\mathfrak{m}_1 + \int_{A_0} f_n d\mathfrak{m}_1 < \int_{A \backslash A_0} f_n d\mathfrak{m}_1 + \epsilon$$

so that

$$\underline{\lim}\int_A f_n d\mathfrak{m}_1 \le \underline{\lim}\int_{A\setminus A_0} f_n d\mathfrak{m}_1 + \underline{\lim}\epsilon.$$

ϵ does not depend on n so

$$\underline{\lim}\int_A f_n d\mathfrak{m}_1 \le \epsilon \Longrightarrow \underline{\lim}\int_A f_n d\mathfrak{m}_1 = 0.$$

■

Theorem 194 *If f is absolutely continuous on $[a,b]$, then $\int_a^b f' d\mathfrak{m}_1 = f(b) - f(a)$*

Proof. Since f is absolutely continuous, it is of bounded variation. By **Corollary 178**, $D_{1/n}f \to f'$ almost everywhere so that

$$\int_a^b f' d\mathfrak{m}_1 = \int_a^b \lim_{n\to\infty}\left(D_{1/n}f\right) d\mathfrak{m}_1 = \lim_{n\to\infty}\int_a^b \left(D_{1/n}f\right) d\mathfrak{m}_1$$

by the previous lemma. Set $h = \frac{1}{n}$. Then,

$$\lim_{h\to 0}\int_a^b (D_h f)\, d\mathfrak{m}_1 = \lim_{h\to 0}\left[\frac{1}{h}\left(hf(b) - \int_a^{a+h} f d\mathfrak{m}_1\right)\right] = \lim_{h\to 0}\left[f(b) - \frac{1}{h}\int_a^{a+h} f d\mathfrak{m}_1\right]$$

where the first equality follows similar line of reasoning as in the proof of **Corollary 168**. The proof can then be completed by observing that since f is continuous,

$$\lim_{h\to 0}\int_a^{a+h} f d\mathfrak{m}_1 = f(a).$$

■

Theorem 195 (Fundamental Theorem of Calculus) *If f is absolutely continuous on $[a,x]$, then $f(x) = f(a) + \int_a^x f' d\mathfrak{m}_1$ for every x.*

Theorem 196 *If f is absolutely continuous on $[a,b]$ if and only if there exists a $g \in \mathcal{L}^1(\mathfrak{m}_1)$ such that $f(x) = f(a) + \int_a^x g d\mathfrak{m}_1$ for every x*

Proof. ($\Longleftarrow$) Take $g = f'$

($\Longrightarrow$) Let $g \in \mathcal{L}^1(\mathfrak{m}_1)$. Then, $|g| \in \mathcal{L}^1(\mathfrak{m}_1)$ so that $\forall\epsilon > 0$, $\exists\delta > 0$ such that

$$\mathfrak{m}_1(A) < \delta \Longrightarrow \int_A g d\mathfrak{m}_1 < \epsilon.$$

If $\{(a_j, b_j) : 1 \le k \le n\}$ is a disjoint collection of intervals, then

$$\sum_{j=1}^n (b_j - a_j) < \delta$$

so that

$$\begin{aligned}\sum_{j=1}^{n}\left|f\left(b_j\right)-f\left(a_j\right)\right| &= \sum_{j=1}^{n}\left|\int_{\left(a_j,b_j\right)} g d\mathfrak{m}_1\right| \\ &\leq \sum_{j=1}^{n}\int_{\left(a_j,b_j\right)}|g|\, d\mathfrak{m}_1 < \epsilon\end{aligned}$$

■

Corollary 197 *If f is increasing on $[a,b]$ and*

$$\int_a^b f' d\mathfrak{m}_1 = f(b) - f(a)$$

then f is absolutely continuous.

Proof. Assume for the sake of contradiction that f is not absolutely continuous. Then, from **Theorem 196**, we may assume that

$$\int_a^x f' d\mathfrak{m}_1 < f(x) - f(a) \text{ but } \int_x^b f' d\mathfrak{m}_1 \leq f(b) - f(x)$$

Then,

$$\int_a^x f' d\mathfrak{m}_1 + \int_x^b f' d\mathfrak{m}_1 < f(b) - f(a)$$

a contradiction. ■

Problem 198 *Let f be absolutely continuous on $[a,b]$ and $f' \equiv 0$ almost everywhere. Prove that f is constant.*

Solution 199 *Let $f(a) = c$ and $x \in [a,b]$. Since f is absolutely continuous on $[a,b]$, we must have*

$$\int_a^x f' d\mathfrak{m}_1 = f(x) - f(a)$$

However, since $f' \equiv 0$ a.e., the LHS of the above equation is zero. Hence $f(x) = f(a) = c$. Since x was arbitrary, we are done.

Proposition 200 *For a summable function f,*

$$f \equiv 0 \text{ a.e.} \iff \int_x^y f d\mathfrak{m}_1 = 0 \ \forall x, y \in [a,b]$$

Proof. Observe that the forward direction ($\Longrightarrow$) is trivial. For the reverse direction,

$$\int_I f d\mathfrak{m}_1 = 0$$

for every I tells us that

$$\int_G f d\mathfrak{m}_1 = 0$$

for any open G so that the integral has the same value for any G_δ set and so, we have the same value for any measurable set and for set of measure 0. Take $E^+ = \{x : f(x) \geq 0\}$. This set is measurable. Then,

$$\int_{E^+} f d\mathfrak{m}_1 = 0$$

but note that

$$\int_{E^+} f d\mathfrak{m}_1 = \int_{[a,b]} f^+ d\mathfrak{m}_1 = 0$$

so that f^+ is zero almost everywhere. Similarly, for $E^- = \{x : f(x) \leq 0\}$. Then,

$$\int_{E^-} f d\mathfrak{m}_1 = 0$$

but note that

$$\int_{E^-} f d\mathfrak{m}_1 = \int_{[a,b]} f^- d\mathfrak{m}_1 = 0$$

so that f^- is zero almost everywhere. Since $f = f^+ - f^-$, we have that f is zero almost everywhere. ■

Theorem 201 *If f is a summable function on $[a,b]$, then for almost every $x \in (a,b)$, we have*

$$\frac{d}{dx} \int_a^x f d\mathfrak{m}_1 = f(x)$$

Proof. Let

$$F(x) = \int_a^x f d\mathfrak{m}_1$$

Then, f is summable so that F is absolutely continuous and $F(a) = 0$. This tells us that

$$F(x) = \int_a^x F' d\mathfrak{m}_1$$

Subtracting both integrals gives us

$$\int_a^x (F' - f) d\mathfrak{m}_1 = 0$$

for every $x \in [a,b]$ so that

$$\int_a^y (F' - f) d\mathfrak{m}_1 = 0 \Longrightarrow \int_x^y (F' - f) d\mathfrak{m}_1 = 0 \text{ for any } x, y$$

$\Longrightarrow F' = f$ almost everywhere by **Proposition 200**. ■

Problem 202 *Let f be continuous on $[a,b]$, differentiable almost everywhere. Then*

$$\int_a^b f' d\mathfrak{m}_1 = f(b) - f(a)$$

if and only if

$$\int_a^b \lim_{n\to\infty} D_{1/n} f d\mathfrak{m}_1 = \lim_{n\to\infty} \int_a^b D_{1/n} f d\mathfrak{m}_1$$

Solution 203 ($\Longrightarrow$) *Referring back to the reasoning in* ***Corollary 168****, we know that*

$$\int_a^b D_h d\mathfrak{m}_1 = \frac{1}{h}\left(\int_b^{b+h} f d\mathfrak{m}_1 - \int_a^{a+h} f d\mathfrak{m}_1\right)$$

so that

$$\begin{aligned}
\lim_{n\to\infty} \int_a^b D_{1/n} d\mathfrak{m}_1 &= \lim_{h\to 0} \int_a^b D_h d\mathfrak{m}_1 \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_b^{b+h} f d\mathfrak{m}_1 - \int_a^{a+h} f d\mathfrak{m}_1\right) \\
&= f(b) - f(a)
\end{aligned}$$

Now, since f is differentiable almost everywhere, then

$$\lim_{n\to\infty} D_{1/n} f = \lim_{n\to\infty} \frac{f(x+1/n) - f(x)}{h} = f'$$

Then,

$$\int_a^b \lim_{n\to\infty} D_{1/n} f d\mathfrak{m}_1 = \int_a^b f' d\mathfrak{m}_1 = f(b) - f(a)$$

Thus,

$$\lim_{n\to\infty} \int_a^b D_{1/n} d\mathfrak{m}_1 = \int_a^b \lim_{n\to\infty} D_{1/n} d\mathfrak{m}_1$$

($\Longleftarrow$) *Since*

$$\int_a^b \lim_{n\to\infty} D_{1/n} d\mathfrak{m}_1 = \int_a^b f' d\mathfrak{m}_1$$

and

$$\begin{aligned}
\lim_{n\to\infty} \int_a^b D_{1/n} d\mathfrak{m}_1 &= \lim_{h\to 0} \int_a^b D_h d\mathfrak{m}_1 \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_b^{b+h} f d\mathfrak{m}_1 - \int_a^{a+h} f d\mathfrak{m}_1\right) \\
&= f(b) - f(a)
\end{aligned}$$

Given that

$$\int_a^b \lim_{n\to\infty} D_{1/n} d\mathfrak{m}_1 = \lim_{n\to\infty} \int_a^b D_{1/n} d\mathfrak{m}_1$$

it follows that

$$\int_a^b f' d\mathfrak{m}_1 = f(b) - f(a)$$

We even the famed product rule!

Problem 204 *Let f and g be absolutely continuous on $[a,b]$. Show that*

$$\int_a^b f.g' d\mathfrak{m}_1 = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f'.g d\mathfrak{m}_1$$

Solution 205 *By **Proposition 183**, $f.g$ is absolutely continuous on $[a,b]$. Recall that $(f.g)' = fg' + f'g$. Also recall that absolute continuity on $[a,b]$ of f (and g) tells us that*

$$\int_a^b f' d\mathfrak{m}_1 = f(b) - f(a) \text{ and } \int_a^b g' d\mathfrak{m}_1 = g(b) - g(a)$$

Then,

$$\int_a^b f'.g d\mathfrak{m}_1 + \int_a^b f.g' d\mathfrak{m}_1 = \int_a^b (f.g)' d\mathfrak{m}_1 = f(b)\,g(b) - f(a)\,g(a)$$

Rearranging this gives us the desired result.

Problem 206 *Let f be strictly increasing and absolutely continuous on $[0,1]$. Show that*

1. *If $G \subset (0,1)$ is open, then*

$$\int_G f' d\mathfrak{m}_1 = \mathfrak{m}_1(f(G))$$

2. *Show the same when G is a G_δ set*
3. *Show the same when $\mathfrak{m}_1(G) = 0$.*
4. *Deduce that the same is true for any measurable set G.*

Solution 207 *1. Since G is open, G is a disjoint union of half-open intervals $\{I_i : i \in \mathbb{N}\}$, where I_i are allowed to be empty for some i and that $\mathfrak{m}_1(I_i) \in [0,1)$ for each i. Thus, we can have*

$$\int_G f' d\mathfrak{m}_1 = \int_{\cup I_i} f' d\mathfrak{m}_1 = \sum_{i=1}^{\infty} \int_{I_i} f' d\mathfrak{m}_1 \tag{9}$$

Let end-points of interval I_i be a_i and b_i, where $a_i \leq b_i$ (half open degenerate intervals are empty). Since f is strictly increasing and absolutely continuous on $[0,1]$, we must have, for each i,

$$\int_{I_i} f' d\mathfrak{m}_1 = f(b_i) - f(a_i).$$

Since

$$\mathfrak{m}_1(G) = \sum_{i=1}^{\infty} \mathfrak{m}_1(I_i) = \sum_{i=1}^{\infty} b_i - a_i.$$

Moreover, since f is increasing and $b_i \geq a_i$, by ***Problem 30****, we must have $\mathfrak{m}_1(f(I_i)) = f(b_i) - f(a_i)$. It follows that*

$$\sum_{i=1}^{\infty} \mathfrak{m}_1(f(I_i)) = \sum_{i=1}^{\infty} f(b_i) - f(a_i) = \mathfrak{m}_1(f(G)).$$

By Eq. (9)*, we are done.*

2.

3.

4. Let E be a measurable set. Then, E can be written as the disjoint union of countable number of open, half-open, degenerate intervals and G_δ sets. Let

$$E = \bigcup_{i=1}^{\infty} E_i \implies f(E) = \bigcup_{i=1}^{\infty} f(E_i)$$

Since f is strictly increasing, $f(E_i)$ is pairwise disjoint. Moreover, if a_i and b_i are (not necessarily distinct) endpoints of E_i, then $\mathfrak{m}_1(f(E_i)) = f(b_i) - f(a_i)$ so that

$$\sum_{i=1}^{\infty} \mathfrak{m}_1(f(E_i)) = \sum_{i=1}^{\infty} f(b_i) - f(a_i) = \mathfrak{m}_1(f(E))$$

It follows that

$$\int_E f' d\mathfrak{m}_1 = \sum_{i=1}^{\infty} \int_{E_i} f' d\mathfrak{m}_1 = \sum_{i=1}^{\infty} \mathfrak{m}_1(f(E_i)) = \mathfrak{m}_1(f(E))$$

As seen above, the notion of absolute continuity is closely tied the the Fundamental Theorem of Calculus for the Lebesgue integral and filters the idea of bounded variation, crucially relying on compatibility with arbitrary but disjoint sums. Can we do better than disjoint sums? Let us agree to call a function $f : [a,b] \longrightarrow \mathbb{R}$ **super absolutely continuous** if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for a finite family of intervals (not necessarily disjoint) $\{(a_i, b_i) : 1 \leq i \leq n\}$, we have

$$\sum_{j=1}^{n} (b_j - a_j) < \delta \Longrightarrow \sum_{j=1}^{n} |f(b_j) - f(a_j)| < \epsilon$$

Is it true that f is absolutely continuous implies that f is super absolutely continuous? No. Let $\epsilon > 0$ and consider $f : [0,1] \longrightarrow [0,1]$ defined by $x \longmapsto \sqrt{x}$. Then, by Archimedean's Principle, $\exists n$ such that $\epsilon > \frac{1}{n}$. Now, consider the family $\{(0, 1/i) : 1 \leq i \leq n\}$. Then,

$$\sum_{j=1}^{n} |f(1/j) - f(0)| = \sum_{j=1}^{n} 1/\sqrt{j} < n \not< \epsilon$$

for any δ. However, $\sqrt{x}$ is absolutely continuous. To show this, observe that $\sqrt{x}$ is increasing on $[0,1]$. Moreover, since f' is continuous on $(0,1)$, it is Riemann Integrable, so that

$$\int_{[0,x]} \frac{1}{2\sqrt{x}} d\mathfrak{m}_1 = \int_0^x \frac{1}{2\sqrt{x}} dx = f(x) - f(0) = \sqrt{x}$$

Hence f is absolutely continuous on $[0,1]$.

4.3 A Pathological Function

When we were building the Cantor set C, at the k-th step, we threw away, in total, $2^k - 1$ open intervals (this includes those thrown away at k-th step!). Let $O_k = I_1^k \cup I_2^k \cup ... \cup I_{2^k-1}^k$ be the union of these open intervals at the k-th step. Note that $O_k \subset O_{k+1}$. Define $\varphi(x) = m/2^k$ for $x \in I_m^k$. To show that φ is well-defined, let us explore some of its values. At $k = 1$, $\varphi(x) = 1/2$ in the middle erased interval and at $k = 2$, $\varphi(x) = 2/2^2$ in the same interval. Moreover, in the second erased interval $(1/9, 2/9)$, $\varphi(x) = 1/4$ and at $(7/9, 8/9)$, $\varphi(x) = 3/4$. Now, to show that φ is indeed well-defined, assume $I \subset O_k$ and $I \subset O_{k+1}$. Then, $I = I_{2m}^{k+1}$ and $I = I_m^k$ so that in O_k, $\varphi(x) = 2m/2^{k+1} = m/2^k$.

φ is increasing on each O_k. Let

$$O = \bigcup_{k=1}^{\infty} O_k$$

and define $\varphi(0) = 0$. Then, $\varphi(x) = \sup\{\varphi(t) : t < x : t \in O, x \in [0,1] \backslash O = C\}$

φ is continuous on O. To show this, note that φ is constant on each open interval. However, φ is not constant on $[0,1]$. Moreover, $\mathfrak{m}_1(O) = 1$.

Let $x_0 \in O$, then $\varphi(x_0)$ is constant around x_0. Assume that $x_0 \in C$ where $1 \neq x_0 \neq 0$. If k is any number, then x_0 lies between I_m^k and I_{m+1}^k. The idea is that if φ is not continuous at x_0, it only has the choice of jumping at x_0 since φ is increasing on a closed and bounded interval. Take $b_k \in I_{m+1}^k$ and $a_k \in I_m^k$. Then, $a_k < x_0 < b_k$ and that $\varphi(a_k) = \frac{m}{2^k}$ and $\varphi(b_k) = \frac{m+1}{2^k}$. Thus, $\varphi(b_k) - \varphi(a_k) = 1/2^k$. As k increases, the jump vanishes: if φ has a jump at x_0, we always have, for some $\delta > 0$, $\varphi(b) - \varphi(a) \geq \delta$ if $a < x_0 < b$ where $\varphi(b) \geq \varphi(x_0^+)$ and $\varphi(a) \leq \varphi(x_0^-)$ so that $\varphi(b) - \varphi(a) \geq \varphi(x_0^+) - \varphi(x_0^-)$, a contradiction.

φ is not absolutely continuous since the fundamental theorem does not hold.

Proof. Let $O = [0,1] \backslash C$. Let us look at $\varphi(O)$. Since

$$O = \bigcup_{k=1}^{\infty} I_k$$

where I_k are disjoint intervals, not necessarily in increasing order. By construction, $\psi|_{I_k}$ is constant for each k. Note that $\varphi(I_k) = I_k + c$, where $\varphi(I_k) = c$. Thus, φ translates I_k and so $\mathfrak{m}_1(\varphi(I_k)) = \mathfrak{m}_1(I_k)$. Moreover, $\{\varphi(I_k) : k \in \mathbb{N}\}$ is a disjoint family since φ is increasing. Thus,

$$\mathfrak{m}_1(\varphi(O)) = \sum_{k=1}^{\infty} \mathfrak{m}_1(\varphi(I_k)) = \sum_{k=1}^{\infty} \mathfrak{m}_1(I_k) = \mathfrak{m}_1(O) = 1$$

Note that $\varphi(O) \cap \varphi(C) = \varnothing$ and that $\mathfrak{m}_1(\varphi([0,1])) = 2$ so that $\mathfrak{m}_1(\varphi(C)) = 2 - \mathfrak{m}_1(\varphi(O)) = 1$. ■

Corollary 208 $\exists$ *a subset of a Cantor set A, measurable by completeness of Lebesgue measure, such that $\varphi(A) \notin \mathfrak{M}_1$.*

Proof. We will use the fact that $[0,1]$ has a non-measurable set. Any subset $E \subset [0,1]$ (not necessarily with finite out measure) with $\mathfrak{m}_1(E) > 0$ contains a non-measurable subset (exercise). Therefore, $\exists B \subset \varphi(C)$ such that $B \notin \mathfrak{M}_1$. Now, define $A = \varphi^{-1}(B)$. Then, $A \subset C$ but $B = \varphi(A) \notin \mathfrak{M}_1$ ■

As a remark, we state without proof that if $A \in \mathfrak{L}_1$ and f is a strictly increasing and continuous function, then $f(A) \in \mathfrak{L}_1$. To show that, one may proceed by observing that A may be split into intervals. We can now show that $\mathfrak{L}_1$ is not complete.

Proof. We need to show that $\exists A \in \mathfrak{M}_1$ but $A \notin \mathfrak{L}_1$. Take $A \subset C$. Then, $\psi(A) \notin \mathfrak{M}_1$. Since $\mathfrak{m}_1(C) = 0$, $A \in \mathfrak{M}_1$. If A was in $\mathfrak{L}_1$, then by previous fact, $\psi(A) \in \mathfrak{L}_1 \subset \mathfrak{M}_1$, a contradiction. Thus, $(\mathbb{R}, \mathfrak{L}_1, \mathfrak{m}_1)$ is not complete. ■

5 Differentiation of Measures

In this section, our big goal is to split our master set X into a positive and negative set, where positive and negative have a different technical name.

5.1 Signed Measures

Definition 209 *Let $\mathfrak{A}$ be a σ-algebra on X, and $\upsilon : \mathfrak{A} \longrightarrow [-\infty, \infty]$. Then, υ is called a* ***signed measure*** *or a* ***charge*** *if*

1. $\upsilon(\varnothing) = 0$
2. *υ assumes either $+\infty$ or $-\infty$ for any $A \in \mathfrak{A}$*

3. If $\{A_k : k \in \mathbb{N}\} \subset \mathfrak{A}$ is a family of disjoint sets, then

$$A = \bigcup_{k=1}^{\infty} A_k \Longrightarrow \upsilon(A) = \sum_{k=1}^{\infty} \upsilon(A_k)$$

and the series converges absolutely if $|\upsilon(A)| < \infty$.

*We call a set $A \in \mathfrak{A}$ **positive** if for every measurable $E \subset A$, we have $\upsilon(E) \geq 0$ and **null** if for every measurable $E \subset A$, we have $\upsilon(E) = 0$*

Example 210 *Assume that we have two signed measures, μ_1, μ_2. Then, $\mu_1 - \mu_2$ is not necessarily subadditive hence not even countably additive. However, it is additive and gives zero at the empty set.*

Example 211 *Let f be a μ-measurable function, then*

$$\upsilon(A) = \int_A f d\mu$$

is a signed measure. Notice that if we let $E^+ = \{x : f(x) \geq 0\}$ and $E^- = \{x : f(x) \leq 0\}$, then $X = E^- \cup E^+$ and

$$\int_{E^+} f d\mu = \int_X f^+ d\mu \text{ and } \int_{E^{--}} f d\mu = \int_X f^- d\mu$$

Proposition 212 *If A is positive, $E \subset A$ is measurable, then E is positive.*

Proof. Obvious ■

Proposition 213 *If $\{A_i : i \in \mathbb{N}\}$ is positive for each i, then so is the union of the family.*

Proof. Let

$$E \subset \bigcup_{k=1}^{\infty} A_k$$

and $E_1 = E \cap A_1$, $E_2 = (E \cap A_2) \setminus E_1, ..., E_n = (E \cap A_n) \setminus (E_1 \cup ... \cup E_{n-1})$ and pairwise disjoint. Then, $\upsilon(E_k) \geq 0$ for each k and

$$\upsilon(E) = \sum_{k=1}^{\infty} \upsilon(E_k) \geq 0$$

■

Lemma 214 (Hahn) *Let $0 < \upsilon(A) < \infty$. Then, there exists a positive set $E \subset A$ with $\upsilon(E) > 0$.*

Proof. If A is positive, we are done, since A has strictly positive measure. If A is not positive, then $\exists \widetilde{A} \subset A, \upsilon\left(\widetilde{A}\right) < 0$. Let

$$m_1 = \min\left\{m \in \mathbb{N} : \exists \widetilde{A} \subset A, \upsilon\left(\widetilde{A}\right) < -1/m\right\}$$

and take A_1 with $\upsilon(A_1) < -1/m_1$. If $A \backslash A_1$ is positive, then we are done. Otherwise, let

$$m_2 = \min\left\{m \in \mathbb{N} : \exists \widetilde{A} \subset A \backslash A_1, \upsilon\left(\widetilde{A}\right) < -1/m_1, \upsilon(A_2) < -1/m_2\right\}$$

Assume that we have $A_1, ..., A_n$. That is, at the n-th step, if

$$A \backslash \bigcup_{k=1}^{n} A_k$$

is not positive, we have

$$m_n = \min\left\{m \in \mathbb{N} : \exists \widetilde{A} \subset A \backslash \bigcup_{k=1}^{n-1} A_k, \upsilon\left(\widetilde{A}\right) < -1/m_1, \upsilon(A_n) < -1/m_n\right\}$$

Now we have $A_{n+1} \subset A \backslash (A_1 \cup A_2 \cup ... \cup A_n)$ with $\upsilon(A_{n+1}) < -1/m_n$. We now claim that

$$E = A \backslash \bigcup_{n=1}^{\infty} A_n$$

is positive. Note that the A is the union of two disjoint sets:

$$A = E \cup \bigcup_{n=1}^{\infty} A_n \implies \upsilon(A) = \upsilon(E) + \upsilon\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$\upsilon(A) < \infty$, then the right hand side is also finite. In particular, υ for the infinite union is finite. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} |\upsilon(A_n)| &< \infty \implies -\infty < \sum_{n=1}^{\infty} \upsilon(A_n) < \sum_{n=1}^{\infty} -\frac{1}{m_n} \\ &\implies \sum_{n=1}^{\infty} \frac{1}{m_n} < \infty \implies \lim_{n\to\infty} m_n = \infty \end{aligned}$$

Now let $B \subset E$. Our goal is to show that $\upsilon(B) \geq 0$. Note that we can have an $N \in \mathbb{N}$ such that

$$B \subset A \backslash \bigcup_{n=1}^{\infty} A_n \subset A \backslash \bigcup_{n=1}^{N} A_n$$

Then, because m_N was the minimum, we have $\upsilon(B) \geq 1/(m_N - 1)$. This is true for every N so we can apply limit and get $\upsilon(B) \geq 0$.

Now, finally

$$v(A) \geq 0 \implies v(A \backslash E) = v(A) - v(E) = v\left(\bigcup_{n=1}^{\infty} A_n\right) < 0$$

so that $v(E) > 0$. ■

Essentially, for any given set X, we can split X into two disjoint sets, one for positive and the other for negative

Theorem 215 (Hahn Decomposition) *Let v be a signed measure on $(X, \mathfrak{A})$. Then, $\exists A \subset X$ with A positive and $\exists B \subset X$ with B negative such that $X = A \cup B$ and $A \cap B = \varnothing$*

Proof. WLOG, we can assume that v is never ∞. If $v(E) \leq 0$ for every $E \subset X$, then set $A = \varnothing$ and $B = X$. If there exists an E with $v(E) > 0$, then, by Hahn's Lemma, we can have a positive subset. Let $\lambda = \sup\{v(E) : E \text{ is positive}\}$. Note that $\lambda \geq 0$. By definition of supremum, there exists a sequence of sets $\{A_k : k \in \mathbb{N}\}$ such that $v(A_k) \to \lambda$ as $k \to \infty$.

Set

$$A = \bigcup_{k=1}^{\infty} A_k$$

since union of positive sets is positive, A is positive. Thus, $\lambda \geq v(A)$. On the other hand, $A \backslash A_k \subset A$ so that $v(A \backslash A_k) \geq 0$. Since both sets have finite measure, we have $v(A) \geq v(A_k)$ for each k. Taking limit on both sides, we get $\lambda \leq v(A)$. Thus, $\lambda = v(A)$.

Now let $B = X \backslash A$. We need to show that B is negative. If not, then there exists an $E \subset B$ with $v(E) > 0$. Thus, there is a positive subset $\widetilde{E} \subset E$ such that $0 < v\left(\widetilde{E}\right)$. Then, $A \cup \widetilde{E}$ is also positive. Moreover, $v(A) + v\left(\widetilde{E}\right) \geq v(A) = \lambda$, a contradiction to the fact that λ is the supremum. Thus, B is indeed negative. ■

We then get the following useful corollary, which we state without proof.

Corollary 216 (Jordan Decomposition) *If v is a signed measure, then there exist two measures ν^+ and ν^- (not signed!) such that $v = v^+ - v^-$ and there exist sets A and B such that $A \cup B = X$, $A \cap B = \varnothing$, $v^+(B) = v^-(A) = 0$.*

Problem 217 *This Jordan Decomposition in unique*

Solution 218 *Assume that $\exists$ measures ν^+ and ν^- (not signed!), positive $A \in \mathfrak{A}$ and negative $B \in \mathfrak{A}$ such that $X = A \cup B$ and $A \cap B = \varnothing$ such that $v = v^+ - v^-$ and $v^+(B) = v^-(A) = 0$. Let $v = \mu^+ - \mu^-$ be another Jordan decomposition of v with positive and negative A' and B'. We first need to show that consideration of different sets is immaterial.*

Consider $A \triangle A' = A \cup A' \backslash (A \cap A')$. Let $E \subset (A \cup A') \backslash (A \cap A')$ be measurable. In particular, $E \subset A \cup A'$. Then, $v(E) \geq 0$. Since $(A \cup A') \backslash (A \cap A') =$

$(A \cup A') \cap (A^c \cup A'^c) = (A \cup A') \cap (B \cup B')$, then, in particular, $E \subset B \cup B'$. Hence $\upsilon(E) \le 0$ so that $\upsilon(E) = 0$. Thus, $A \triangle A'$ is a null-set with respect to υ. Similarly, $B \triangle B'$ is a null set with respect to υ.

Now let us focus on $\upsilon = \mu^+ - \mu^-$. By above, we can work with same A and B. Let $E \in \mathfrak{A}$. Then, $E \cap A \subset A$ so that $\upsilon(E \cap A) = \upsilon^+(E \cap A) - \upsilon^-(E \cap A)$. Since $E \cap A$ is positive, $\upsilon^-(E \cap A) = 0$. Thus, $\upsilon(E \cap A) = \upsilon^+(E \cap A)$. Now, since υ^+ is a measure and A is υ^+-measurable, for any set E, $\upsilon^+(E) = \upsilon^+(E \cap A) + \upsilon^+(E \cap A^c) = \upsilon^+(E \cap A) + \upsilon^+(E \cap B)$. Since $E \cap B$ is negative, $\upsilon^+(E \cap B) = 0$.

Thus, $\upsilon^+(E) = \upsilon^+(E \cap A) = \upsilon(E \cap A)$. Since $\upsilon(A \triangle A') = 0$, we must have $\upsilon(E \cap A) = \upsilon(E \cap A')$. Also since $\upsilon = \mu^+ - \mu^-$, we can have $\upsilon(E \cap A') = (\mu^+ - \mu^-)(E \cap A') = \mu^+(E \cap A') - \mu^-(E \cap A')$. Since $E \cap A'$ is positive, $\mu^-(E \cap A') = 0$. Hence $\upsilon(E \cap A') = \mu^+(E \cap A')$.

Thus far, we have that $\upsilon^+(E) = \upsilon^+(E \cap A) = \mu^+(E \cap A')$. Now, again, $\mu^+(E) = \mu^+(E \cap A') + \mu^+(E \cap A'^c) = \mu^+(E \cap A') + \mu^+(E \cap B') = \mu^+(E \cap A') + 0$.

In summary, for any E, $\upsilon^+(E) = \mu^+(E)$. Hence $\upsilon^+ = \mu^+$. Combining this with the assumption that $\upsilon^+ - \upsilon^- = \mu^+ - \mu^-$ gives us $\upsilon^- = \mu^-$. Thus, the Jordan decomposition is unique.

Problem 219 *For a signed measure, define*

$$|v|(E) = v^+(E) + v^-(E).$$

Show that $|v|(E)$ is equal to

$$\sup \sum_{k=1}^{n} |v(E_k)|$$

where supremum is taken over all finite disjoint families $\{E_k : 1 \le k \le n\}$ of measurable subsets of X.

Solution 220 *Let $E \in \mathfrak{A}$. Then, $v(E) = v^+(E) - v^-(E) \le v^+(E) \le v^+(E) + v^-(E) = |v|(E)$. Conversely, $-v(E) = v^-(E) - v^+(E) \le v^-(E) \le v^+(E) + v^-(E) = |v|(E)$. That is, $-|v|(E) \le v(E)$. Together, these imply that $|v(E)| \le |v|(E)$ for any measurable E. Let $\{E_k : 1 \le k \le n\}$ be a disjoint family of E. Then, note that*

$$|v|(E) = v^+(E) + v^-(E) = \sum_{k=1}^{n} v^+(E_k) + \sum_{k=1}^{n} v^-(E_k) = \sum_{k=1}^{n} |v|(E_k)$$

Moreover, we have that $|v(E_k)| \le |v|(E_k)\ \forall k$. Thus,

$$\sum_{k=1}^{n} |v(E_k)| \le \sum_{k=1}^{n} |v|(E_k) = |v|(E)$$

Since this holds for an arbitrary decomposition of E, we must have

$$\sup \sum_{k=1}^{n} |v(E_k)| \leq |v|(E)$$

On the other hand, by Hahn Decomposition, $\exists$ positive A and negative B such that $X = A \cup B$. Then, $|v|(X) = v^+(A) + v^-(B) = |v^+(A)| + |v^-(B)|$ with $A \cap B = \varnothing$. Now, note that for any measurable E can be partitioned using these positive and negative sets (which is a candidate for the supremum) since $E = E \cap X = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$ and that $|v|(E) = v^+(E) + v^-(E) = v^+(E \cap A) + v^-(E \cap A) + v^+(E \cap B) + v^-(E \cap B)$

$= v^+(E \cap A) + v^-(E \cap B) \leq v(E \cap A) + v(E \cap B) \leq |v(E \cap A)| + |v(E \cap B)|$. Thus, the other side of the inequality holds.

Assuming that f is positive and measurable and that

$$\int_E f d\mu = 0$$

whenever $\mu(E) = 0$, then for any measurable set A, we can define the measure

$$v(A) = \int_A f d\mu$$

Thus, if v and μ are defined on $(X, \mathfrak{A})$, we say that v is **absolutely continuous with respect to** μ if $\mu(E) = 0 \implies v(E) = 0$. The notation in this case is $v \ll \mu$.

Theorem 221 *Assume that $v(X) < \infty$. Then, $v \ll \mu \iff \forall \epsilon > 0, \exists \delta > 0 : \forall E \in \mathfrak{A}, \mu(E) < \delta \implies v(E) < \epsilon$*

Proof. ($\Longleftarrow$) Fix any ϵ; take E with $\mu(E) = 0$. Then, $\mu(E) = 0 < \delta \implies v(E) < \epsilon$. Since this is true for any ϵ, then $v(E) = 0$.

($\implies$) For the sake of contradiction, assume that the conclusion fails. Then, $\exists \epsilon_0 > 0 : \forall \delta > 0, \exists E \in \mathfrak{A}$, which depends on δ, $\mu(E) < \delta$ but $v(E) \geq \epsilon$.

Let E_n be subsets of E such that $\mu(E_n) = 1/2^n$ and let

$$A_N = \bigcup_{k \geq N} E_k \implies \mu(A_N) \leq \sum_{k=N}^{\infty} \frac{1}{2^k} = \frac{1}{2^{N-1}}$$

Now let

$$A = \bigcap_{N=1}^{\infty} A_N$$

Then, $\mu(A) \leq \mu(A_N) \leq 1/2^{N-1}$ for every N so that $\mu(A) = 0$. However, $v(A) > 0$: to see this, note that $v(A_N) \geq v(E_N) \geq \epsilon_0$. Since $v(A_1) \leq v(X) < \infty$, then $v(A) = \lim_{N \to \infty} v(A_N) \geq \epsilon_0$. That is, $v(A) > 0$. ■

Example 222 *Take $\mu = \mathfrak{m}_1$ and $\upsilon = \delta_0$ measure. The latter works like a charge at* 0*. Then, it is not true that $\upsilon \ll \mu$ because $\mu(\{0\}) = 0$ but $\upsilon(\{0\}) = 1$.*

Theorem 223 (Radon-Nikodym) *If υ, μ are both σ-finite measures on the same σ-algebra $\mathfrak{A}$ and $\upsilon \ll \mu$, then $\exists$ a nonnegative, measurable (not necessarily summable) function f such that $\upsilon(E) = \int_E f d\mu$.*

Proof. Assume μ, υ are both finite. Also assume that $\upsilon(X) > 0$, for otherwise $f \equiv 0$ works and gives us the trivial case. For $t > 0$, denote $\lambda_t = \upsilon - t\mu$. λ_t is a signed measure but not necessarily nonnegative. We can have sets P_t and N_t which are, respectively, positive for λ_t and negative for λ_t with $P_t \cap N_t = \varnothing$ and $P_t \cup N_t = X$ by the Hahn Decomposition. We need to fix t. We claim that $\exists t$ with $\mu(P_t) > 0$. Assume that is not true. Then, for $\forall E \in \mathfrak{A}$, we have $\lambda_t(E) \leq 0$ since $\lambda_t(E) = \lambda_t(E \cap P_t) + \lambda_t(E \cap P_t^c) = \lambda_t(E \cap P_t) + \lambda_t(E \cap N_t)$
$= \lambda_t(E \cap P_t) = (\upsilon - t\mu)(E \cap P_t) = \upsilon(E \cap P_t) - t\mu(E \cap P_t) \leq 0$. Thus, $\upsilon(X) \leq t\mu(X)$ for every t. But then, $\upsilon(X) = 0$, a contradiction.

Now, define the family

$$\mathcal{F} = \left\{ f \geq 0 : \int_E f d\mu \leq \upsilon(E) \, \forall E \in \mathfrak{A} \right\}$$

This set is non-empty since $f \equiv 0 \in \mathcal{F}$. Is there a non-trivial function? Define $f = t.\chi_{P_t}$ where $\mu(P_t) > 0$. Then,

$$\int_E f d\mu = t\mu(E \cap P_t)$$

Since $\lambda_t(E \cap P_t) \geq 0$, we must have $t\mu(E \cap P_t) \leq \upsilon(E \cap P_t) \leq \upsilon(E)$

We need to find the "biggest" possible f. Let us consider the "average".

$$M = \sup_{f \in \mathcal{F}} \int_X f d\mu$$

Then, $M > 0$. Our next goal is to show that the supremum is actually obtained by showing that $\exists f \in \mathcal{F}$ with

$$M = \int_X f d\mu$$

Observe that if $f, g \in \mathcal{F}$, then $h = \max(f, g) \in \mathcal{F}$. To show this, if $E \in \mathfrak{A}$, take $E_1 = \{x \in E : f \geq g\}$ and $E_2 = \{x \in E : f < g\}$. Then,

$$\int_E h d\mu = \int_{E_1} h d\mu + \int_{E_2} h d\mu = \int_{E_1} f d\mu + \int_{E_2} g d\mu \leq \upsilon(E_1) + \upsilon(E_2) \leq \upsilon(E)$$

Now let $\{f_n : n \in \mathbb{N}\} \subset \mathcal{F}$ such that

$$\int_X f_n d\mu \to M$$

Take $g_n = \max(f_1, ..., f_n)$. Then, g_n is an increasing sequence and $g(x) = \lim_{n\to\infty} g_n(x)$ is well-defined. By Monotone Convergence Theorem,

$$\int_E g d\mu = \lim_{n\to\infty} \int_E g_n d\mu$$

Since each $g_n \in \mathcal{F}$, then, for each n, the left hand side is less than the limit of $v(E)$. That is, $v(E)$ itself. On the other hand,

$$\int_X g d\mu \geq \int_X g_n d\mu \to M \implies \int_X g d\mu = M$$

Now assume that for the measure space $(X, \mathfrak{A})$, μ and ν are σ-finite on X. Then, $\exists \{X_n : n \in \mathbb{N}\} \subset \mathfrak{A}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

where $\mathfrak{m}_1(X_n) < \infty$ for each n. WLOG, we can assume that $X_n \subset X_{n+1}$. Restricting μ on the subspace σ-algebra $\mathfrak{A}_{X_n} = \{E \cap X_n : E \in \mathfrak{A}\}$ gives us a finite measures $\mu|_{\mathfrak{A}_{X_n}} := \mu_n$ and $\nu|_{\mathfrak{A}_{X_n}} := \nu_n$ on σ_{X_n} for each n since $\mathfrak{m}_1(X_n) < \infty$.

This enables us to use the previous case of ν and μ being finite measures; for each n, we can find a unique function f_n measurable with respect to $\mathfrak{A}_{X_n}$ such that

$$\nu_n(E) = \int_E f_n d\mu_n \text{ for all } E \in \mathfrak{A}_{X_n}$$

Now, if $n \leq m$, then $X_n \subset X_m$ so that $\mathfrak{A}_{X_n} \subset \mathfrak{A}_{X_m}$. Thus, μ_n is the restriction of μ_m such that $\mu_n = \mu_m$ for $E \in \mathfrak{A}_{X_n}$. That is, μ_n can be extended to μ_m. It follows that

$$\text{for all } E \in \mathfrak{A}_{X_n},\ \nu_n(E) = \int_E f_n d\mu_n = \int_E f_m d\mu_m = \nu_m(E)$$

so that ν_n is the restriction of ν_m. Since f_n is unique, $f_n = f_m$ for all $X_n \backslash A$ where $\mu_n(A) = 0$. Therefore, if $F \in \mathfrak{A}$, then

$$\nu_n(F \cap X_n) = \int_F f_n d\mu_n \text{ for all } F \in \mathfrak{A}$$

Note that $F \cap X_n \subset F \cap X_{n+1}$ since $X_n \subset X_{n+1}$ and that

$$F = F \cap X = F \cap \bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} (F \cap X_n)$$

Thus, by continuity from above,

$$\nu(F) = \nu\left(\bigcup_{i=1}^{\infty} (F \cap X_n)\right) = \lim_{n\to\infty} \nu_n(F \cap X_n)$$

■

Let $g_n = \max\{f_1, ..., f_n\}$. Then, $\{g_n : n \in \mathbb{N}\}$ is a monotone increasing sequence of sets in X. Let $g = \lim_{n\to\infty} g_n$ (we don't necessarily need g to be summable). Since for each n, g_n is μ-measurable, then g is also μ-measurable. Moreover, by Monotone Convergence Theorem,

$$\int_{F\cap X_n} \lim_{n\to\infty} g_n d\mu_n = \lim_{n\to\infty} \int_{F\cap X_n} g_n d\mu_n = \int_F g d\mu$$

so that there exists a μ-measurable function g such that, for any $F \in \mathfrak{A}$

$$\nu(F) = \int_F g d\mu$$

Problem 224 *(a) Characterize the measure spaces $(X, \mathfrak{A}, \mu)$ for which the counting measure $\mathfrak{M}$ is absolutely continuous with respect to μ; and (b) characterize the measure spaces for which, given $x \in X$, the measure δ_x is absolutely continuous with respect to μ.*

Solution 225 *(a) We know that for any $E \in \mathfrak{A}$, $\mu(E) = 0 \Longrightarrow \mathfrak{M}(E) = 0$. In other words, $\mathfrak{M}(E) \neq 0 \Longrightarrow \mu(E) \neq 0$. If $\mathfrak{M}(E) \neq 0$, then E is non-empty. Since $\mathfrak{M}$ is defined on 2^X, we must have μ defined on $\mathfrak{A} = 2^X$ with $\mu(E) \neq 0$ (that is, $\mu(E) > 0$) for any non-empty set E. Thus the spaces $(X, \mathfrak{A}, \mu)$ we are looking for are precisely those for which there are no non-empty sets of measure 0.*

(b) Let $x \in X$ be fixed. Then, for any $E \in \mathfrak{A}$, $\mu(E) = 0 \Longrightarrow \delta_x(E) = 0$. In other words, $\delta_x(E) \neq 0 \Longrightarrow \mu(E) \neq 0$. That is, $\delta_x(E) = 1 \Longrightarrow \mu(E) > 0$. That is, $x \in E \Longrightarrow \mu(E) > 0$. Thus, given $x \in X$, the spaces $(X, \mathfrak{A}, \mu)$ we are looking for are precisely those for which there are no non-empty sets containing x of measure 0.